Millisecond and Binary Pulsars as Nature's Frequency Standards.

II. Effects of Low-Frequency Timing Noise on Residuals and Measured Parameters

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ABSTRACT

Millisecond and binary pulsars are the most stable natural frequency standards. They can be applied to a number of principal problems of modern astronomy and time-keeping metrology including the search for stochastic gravitational wave background in the early universe, testing General Relativity and establishing of new ephemeris time scale. The full exploration of pulsar properties requires obtaining proper unbiased estimates of the spin and orbital parameters, a problem which deserves a special investigation. These estimates depend essentially on the random noise component being revealed in the residuals of time of arrivals (TOA) and having a different physical origin. In the present paper, the influence of low-frequency ("red") timing noise with spectral indices from 1 to 6 on TOA residuals, variances, and covariances of estimates of measured parameters of single and binary pulsars are studied. In order to determine their functional dependence on time, an analytical technique of processing of observational data in time domain is developed. Data processing in time domain is more informative since it takes into account both stationary component of noise and its non-stationary counterpart. Data processing in frequency domain is valid if and only if the noise is stationary. Our analysis includes a simplified timing model of a binary pulsar in a circular orbit and procedure of estimation of pulsar parameters and residuals under the influence of red noise. We reconfirm, in accordance with results of previous authors, that uncorrelated white noise of errors of measurements of TOA brings on gradually decreasing residuals, variances and covariances of all parameters. On the other hand, we show that any low-frequency, correlated noise of terrestrial or/and astrophysical origin present causes the residuals, variances, and covariances of certain parameters to increase with time. Hence, the low frequency noise corrupts our observations and reduces experimental possibilities for better tests of General Relativity Theory. At the same time, the rate of growth of residuals and variances of parameters can give a valuable information about the red noise itself. We also treat in detail the influence of a polynomial drift of noise on the residuals and fitting parameters in order to avoid confusion with red noise without the polynomial drift. Results of the analitic analysis are used for discussion of a statistic describing stabilities of kinematic (PT) and dynamic (BPT) pulsar time scales.

Key words: methods: data analysis - methods: statistical - pulsars: general, binary

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1 INTRODUCTION

Timing observations of single and, especially, binary millisecond pulsars are widely recognized as being extremely important for progressing a number of branches of modern astronomy and time keeping metrology. In particular, implication of pulsar timing for testing General Relativity in the strong-field regime (Taylor & Weisberg 1982, 1989), creation of a new astronomical time scale based on the high-stable rotation of millisecond pulsars (Ilyasov *et al.* 1989, Kaspi *et al.* 1994)as well as setting up the upper limit on the energy density of stochastic gravitational wave background in early universe (McHugh *et al.* 1996, Kopeikin 1997a, and references therein) are among the most successful and stimulating recent achivements in this area of research.

The accuracy of pulsar timing observations is now approaching ~ 100 nanoseconds. Such high precision requires construction of adequate data processing algorithms taking proper account of the relevant physical effects which can contribute to the timing signal being processed. Significant theoretical understanding of relativistic celestial mechanics of binary systems has been made since the discovery by Hulse and Taylor (1975) of the first binary pulsar PSR B1913+16 (Damour *et al.* 1989, Damour & Schäfer 1988, Ohta & Kimura 1988, Schäfer & Wex 1993, Kopeikin & Potapov 1994, Schäfer 1995, Wex 1995). Presently, one is able to predict both secular evolution and periodic perturbations of the binary system's orbit up to the $2\frac{1}{2}$ post-Newtonian approximation (PNA) where emission of gravitational waves starts to influence the orbital dynamics (Grishchuk & Kopeikin 1983, Damour 1983). Presently, efforts are being made towards developing theory of motion of binary systems in 3 PNA and $3\frac{1}{2}$ PNA of General Relativity (Iyer & Will 1995, Jaranowski & Schäfer 1997, Jaranowski 1997, Rieth 1997).

It is worth remembering that processing of pulsar timing data comprised of topocentric time of arrivals (TOA) of pulsar pulses is based on the χ^2 minimization procedure of fitting observational parameters of the pulsar to the adopted model of the observed TOA. This includes polar motion corrections UT1-UTC given by the International Earth Rotation Service (IERS), the general relativistic model of motion of the observer around the barycenter of the Solar system (Standish 1990) and the pulsar around the barycenter of the binary system (Damour *et al.* 1989), the post-Newtonian transformations between different time scales (Brumberg Kopeikin 1990, Fukushima 1995) used in the model, and the law of propagation of electromagnetic signals in gravitational fields (Shapiro 1964), as well as the interstellar and interplanetary medium (Rickett 1990, 1996). The overall model is rather sophisticated and presently exists in the form of two independently developed computer programs: TEMPO (Taylor & Weisberg 1989), and TIMAPR (Doroshenko & Kopeikin 1990, 1995), both being available on the World Wide Web. It is worthwhile pointing out that TIMAPR is based on the unique theoretical approach for construction of relativistic time scales and reference frames in the Solar system developed in (Kopeikin 1988, Brumberg & Kopeikin 1989, 1990).

Usually, the procedure for estimating pulsar parameters is based on the premise that white noise dominates in TOA residuals. However, long-term monitoring of certain pulsars has revealed the presence of non-white component of noise of astrophysical origin as well (Cordes & Downs 1985, Kaspi *et al.* 1994, D'Allesandro *et al.* 1996). It is customary call such correlated noise as a coloured Gaussian (or, simply, as "red") one because its spectrum diverges at zero frequency ("the

infra-red catastrophe"). At low frequencies the red noise has a non-flat spectrum and can be described in framework of single- or multiple- component power-law model. The lower the timing activity of the pulsar, the further toward low frequencies one must look in order to detect the red noise in the spectrum of its TOA residuals. Although being fairly difficult for detection, red noise contains invaluable information about diversity of physical processes which take place in the neutron star interior, the interstellar medium, early universe, and man-made terrestrial clocks (Lorimer 1996). For this reason, developing a rigorous procedure for accounting for the red noise component in TOA residuals is worthwhile. Physicists have been working on the problem of adequate treatment of red noise for a long time (see, for example, Stratonovich & Sosulin 1966, Van Trees 1968, Hooge 1976, Planat *et al.* 1996). However, one of the first attempts in developing proper statistical procedure of fitting pulsar parameters in the presence of low-frequency, red noise has been undertaken only recently by Bertotti *et al.* (1983), and Blandford *et al.* (1984) in order to obtain unbiased estimates of the pulsar's parameters as well as upper limit on the energy density of the GWB radiation. There is a need to further improve on probabilistic models of the red noise and methods of estimation of its characteristics containing essential information about long-term processes having both geophysical and astrophysical origins, including the stochastic background of primordial gravitational radiation. The present state-of-the art of statistical analysis of pulsar timing observational data in the presence of red noise has not yet reached the required level of clarity and completeness, and more elaborate methodology has to be developed. This point has been especially stressed by Bertotti *et al.* (1983), Hogan & Rees (1984), Blandford *et al.* (1984), and Kopeikin (1997a, 1997b).

Systematic studies of timing noise in post-fit residuals was started by Groth (1975) and Cordes (1978, 1980) (see also Cordes & Helfand (1980), Cordes & Greenstein (1981), and Cordes & Downs (1985)), who put forward a random walk noise model along with the use of orthonormal polynomials for analyzing properties of the noise in the time domain. These investigations stimulated development of a somewhat equivalent approach for estimating the spectral power of the low-frequency noise which was evaluated in the Fourier frequency domain and non-uniformly sampled data by Deeter & Boynton (1982) and Deeter (1984). The power spectrum technique was later applied to pulsar timing data by Deeter *et al.* (1989). This formalism was worked further into a practical procedure by Stinebring *et al.* (1989) and extensively employed by Kaspi *et al.* (1994), and Thorsett & Dewey (1996) for setting an upper limit on the cosmological parameter Ω_g using the Neyman-Pearson test of statistical hypothesis. McHugh *et al.* (1996) recently refined this procedure to make it even more rigorous using the approach based on the Bayesian statistic.

It is worth noting that the analysis in the time domain is more informative than that in the frequency domain. This is because noise contains usually both stationary and non-stationary components, and spectral analysis of the noise in frequency domain is adequate if and only if the noise itself is stationary (or its increments). For this reason, the procedure advanced by Stinebring *et al.* (1989) is principally restricted by the implicit assumption on stationarity of timing noise and, therefore, its implications for processing real observational data are jeopardized. Fortunately, we have been able to prove (Kopeikin 1997a) and reconfirm in the present paper that under rather general circumstances the procedure of fitting of pulsar spin and orbital parameters acts as a low-frequency filter of the non-stationary component of noise which means that pulsar post-fit residuals bear, in fact, only the stationary component of noise and the spectral analysis in the Fourier domain is

legitimate. Nevertheless, non-stationary part of noise intrudes into observed values of pulsar spin-down parameters, increases their variances, and bringing on large correlations with other parameters.

Another thing to notice is that inherent to Deeter-Boynton's procedure are certain limitations, which must be clearly understood and accounted for in practical implementations. The foremost restriction is that it provides a satisfactory guide to the selection as well as construction of practical estimators of a power spectrum of noise as long as the noise is approximately represented by a (2r)-th power law; that is, then it is possible to write the noise spectrum down as $S(f) = K_{2r} f^{-2r}$, where r $= 1, 2, 3, ...$, and any of K_{2r} is a constant. Let us note that the noise in question can be treated as a random walk in rotational phase, frequency, or frequency derivative of the pulsar. Stationary stochastic processes possessing power spectra with odd spectral indices $S(f) = K_{2r+1} f^{-2r-1}$, (r $= 0, 1, 2, ...$) are called flicker noise and require development of a more advanced statistical approach. A possible way towards the development of such an approach in the frequency domain was explored by Blandford *et al.* (1984). However, it is not as far-going as one would desire. From the mathematical point of view the difficulty is due to the existence of an algebraic singularity in the spectrum of low-frequency noise as frequency approaches the point $f = 0$. When the spectral index of the noise is large enough, the singular behavior of the power spectrum leads to formally divergent integrals describing timing residuals and variances of measured parameters. To avoid this problem, a special regularization procedure must be applied to these integrals to ascribe them definite numerical values and, as a consequence, a real physical meaning. There are several known methods of dealing with regularization of functionals with algebraic singularities. Blandford *et al.* (1984) used the simplest procedure of regularization based on the effective spectral cutoff of the divergent integrals. As a consequence, their estimators of spectral power of the noise as well as variances of pulsar parameters depend on the lower cut-off frequency, $f_{\min}$, and two numbers, A, and, N^*, which are used for minimizing root-mean-square error in rotational frequency through the post-fit residuals. The cutoff frequency $f_{\min}$ and numbers A and N^* are not constant and depend on the total span of observations, making temporal behavior of the post-fit residuals and variances of parameters rather uncertain. In addition, the simple cutoff of divergent integrals leads to enormous leakage of the spectral power from the low-frequency tails of the estimators and as a consequence to the wrong determination of the magnitude K_m of the noise power spectrum.

We have suggested tackling the problem of divergent integrals using the theory of generalized functions (Gel'fand & Shilov 1964) as the most powerful and simple in dealing with functions having singular spectra. For instance, this theory has been successfully applied in Quantum Renormalization Theory (Damour 1975) and for calculations of high-order relativistic equations of motion of binary pulsars (Damour 1983a, Schäfer 1985). Quite recently the theory of generalized functions has been applied by Kopeikin (1997a, 1997b) for development of an adequate treatment of low-frequency timing noises with negative integer spectral indices. In particular, we have discovered in these papers for the first time that flicker noise caused by the stochastic background of primordial gravitational radiation in early universe leads to the appearance of specific logarithmic terms in the autocovariance functions as well as in variances of spin parameters of observed pulsars. This result is re-confirmed and considerably extended in the present paper accounting for flicker noise in rotational phase and frequency as well.

As a final critical remark, let us point out that the problem of proper estimation of variances of orbital parameters of

binary pulsars in the presence of low-frequency noise has not been yet discussed in full detail. Therefore, the main goal of the present paper is to provide a self-consistent theoretical and numerical study of this problem. Here we use the results of our previous work (Kopeikin 1997a) in which the general statistical model of red noise has beeen developed. The plan for the present paper is as follows. In the next section we describe an analytical timing model which is used in the subsequent discussion. The procedure of estimation of variances of measured parameters in the presence of low-frequency red noise is outlined in section 3. A brief description of model of a red noise and its autocovariance function are given in section 4. Section 5 explains computational aspects of our algorithm. Polynomial drift of timing noise is discussed in section 6. The drift-free noise model is employed in section 7 for analytical evaluation of variances of spin and orbital parameters of a binary pulsar in the presence of white and the low-frequency noises. Finally, the analytic results are used for studying stability of kinematic (PT) and dynamic (BPT) pulsar time scales in section 8.

To complete the analysis the study of spectral sensitivity of binary pulsars in different frequency bands of the noise spectrum has to be done. This requires more work which is currently in progress. These results will be published elsewhere.

2 TIMING MODEL

We consider a simplified, but still realistic model of arrival time measurements of pulses from a pulsar in a binary system. It is assumed that the orbit is circular, and the pulsar rotates around its own axis with angular frequency ν_p which slows down due to the electromagnetic (or whatever) energy losses. It is also taken into account that the orbital frequency of the binary system, n_b, and its projected semimajor axis, x, have a secular drift caused by emission of gravitational waves from the binary (Peters & Mathews 1963, Peters 1964) bringing about the gravitational radiation reaction force (Damour 1983a, Grishchuk & Kopeikin 1983), radial acceleration (Damour & Taylor 1991, Bell & Bailes 1996), and proper motion of the binary in the sky (Kopeikin 1996).

The moment, $\mathcal{T}$, of emission of the $\mathcal{N}$-th pulsar's pulse relates to the moment, t, of its arrival, measured at the infinite electromagnetic frequency, by the equations (Damour & Taylor 1992, Kopeikin 1994):

$$D\left[\mathcal{T} + x\sin\left(n_b\mathcal{T} + \sigma\right)\right] = t + \phi_0(t) + \phi_1(t), \tag{1}$$

$$t = \tau^* + \Delta_C + \Delta_{R\odot} + \Delta_{\pi\odot} + \Delta_{E\odot} + \Delta_{S\odot}. \tag{2}$$

We use the following notations:

- $\mathcal{T}$ - pulsar time scale,
- t - barycentric time at the barycenter of the Solar system,
- τ^* - topocentric time of observer,
- Δ_C, $\Delta_{R\odot}$, $\Delta_{\pi\odot}$, $\Delta_{E\odot}$, $\Delta_{S\odot}$ - clock and astrometric corrections (Taylor & Weisberg 1989, Doroshenko & Kopeikin 1990, 1995) which one assumes to be known precisely,

- D - Doppler factor gradually changing due to the acceleration and proper motion of the binary system in the sky [†]

- σ - initial (constant) orbital phase,

- n_b - orbital frequency ($n_b = 2\pi/P_b$),

- i - angle of inclination of the orbit to the line of sight,

- x - projected semimajor axis a_p of the pulsar's orbit ($x = a_p \sin i/c$),

- c - speed of light,

- $\phi_0(t)$ - the gaussian noise of TOA measuring errors,

- $\phi_1(t)$ - low-frequency gaussian noise caused by the long-term instabilities of terrestrial clocks, effects in propagation of

radio signals in the interstellar medium, and stochastic background of primordial gravitational waves.

We also suppose that observations start at the time $t_0 = 0$. Changing of the starting time of observations from $t_0 = 0$ to $t_0 \neq 0$ is achieved by the simple translation of time scale: $t \mapsto t - t_0$. It is worth emphasizing that for nearly circular orbits the initial orbital phase $\sigma = \omega_0 - n_b \mathcal{T}_0$ where $\mathcal{T}_0$ is a fiducial moment of time and ω_0 is the longitude of periastron. This relationship remains valid up to the first order of magnitude in eccentricity and represents an excellent approximation in the case of binary pulsars like PSR B1855+09 or PSR J1713+0747.

The rotational phase of the pulsar is given by the polynomial in time

$$\mathcal{N}(t) = \nu_p \mathcal{T} + \frac{1}{2}\,\dot{\nu}_p\,\mathcal{T}^2 + \frac{1}{6}\,\ddot{\nu}_p\,\mathcal{T}^3 + \frac{1}{24}\,\dddot{\nu}_p\,\mathcal{T}^4 + \frac{1}{120}\,\ddddot{\nu}_p\,\mathcal{T}^5 + \nu_p \phi_2(\mathcal{T}) + O[\mathcal{T}^6], \tag{3}$$

where $\nu_p, \dot{\nu}_p, \ddot{\nu}_p, etc.$ are pulsar's rotational frequency and its time derivatives all referred to the epoch $\mathcal{T} = 0$, the term $O[\mathcal{T}^6]$ denotes high order derivatives of the rotational phase, and $\phi_2(\mathcal{T})$ is the intrinsic pulsar timing noise in either rotational phase, frequency, or frequency derivative. Solving iteratively equation (1) with respect to $\mathcal{T}$ and substituting $\mathcal{T}$ for the right hand side of equation (3) gives a relationship between two observable quantities $\mathcal{N}$ and t:

$$\mathcal{N}(t) \;\; = \;\; \mathcal{N}_0 + \nu t + \frac{1}{2}\,\dot{\nu}\,t^2 + \frac{1}{6}\,\ddot{\nu}\,t^3 + \frac{1}{24}\,\dddot{\nu}\,t^4 \tag{4}$$

$$+ \frac{1}{120}\,\ddddot{\nu}\,t^5 - \nu(x + \dot{x}\,t + \frac{1}{2}\,\ddot{x}\,t^2 + \frac{1}{6}\,\dddot{x}\,t^3)\sin(\sigma + n_b t + \frac{1}{2}\dot{n}_b t^2 + \frac{1}{6}\ddot{n}_b t^3) + \nu\,\epsilon(t)\;,$$

with

$$\epsilon(t) \;\; \overset{def}{=} \;\; \phi_0(t) + \phi_1(t) + \phi_2(t)\;, \tag{5}$$

where $\mathcal{N}_0$ is the initial rotational phase of the pulsar ($\mathcal{N}_0 \simeq -\nu t_0$); $\nu, \dot{\nu}, \ddot{\nu}, ...$ are the pulsar's rotational frequency and its time derivatives at the initial epoch t_0; $x, \dot{x}, \ddot{x}, ...$ are the projected semimajor axis of the orbit and its time derivatives at the epoch t_0; $\sigma, n_b, \dot{n}_b, \ddot{n}_b, ...$ are the pulsar's orbital initial phase, orbital frequency and its time derivatives at the epoch t_0. All non-linear terms of order $x^2, x\epsilon, \epsilon^2, \dot{\nu}x, \ddot{\nu}x$, etc. are negligible and have been omitted from (4).

[†] $D = \dfrac{1 + \frac{V_R}{c}}{\sqrt{\left(1 - \frac{V^2}{c^2}\right)}}$, where V_R and V are correspondingly the relative radial and total velcities of the binary system barycenter with

respect to the barycenter of the Solar system

Let us underline that secular variations (time derivatives) of pulsar parameters include not only contributions caused by the dissipative physical mechanisms like emission of gravitational waves by the binary pulsar, but also depend on the kinematical effects caused by the radial acceleration and proper motion of the pulsar in the sky (Damour & Taylor 1991, Bell & Bailes 1996, Kopeikin 1994, 1996).

Another important point which is usually never stressed, but rather crucial, is the treatment of pulsar timing noise $\phi_2(\mathcal{T})$. The noise in question contributes to random jumps of the pulsar's phase at the moment $\mathcal{T}$ of emission of pulse. However, observer at the Earth receives the pulse, and consequently, information about the noise function ϕ_2 much later after the radio pulse has moved across the distance separating the pulsar from observer. Thus, the observer is allowed to think about the pulsar timing noise as starting long time ago and the question arises about how to account for this nuisance "memory" effect. Actually, the "memory" effect problem is not a specific of the timing noise only. It exists in treatment of observations being corrupted by any low frequency noise. As far as we know the only paper, where the "memory" effect in pulsar timing data processing has been discussed, is (Kopeikin 1997b). Therein, we have proved that the "memory" effect is negligibly small under normal circumstances. Namely for this reason we have deliberately omitted any explicit dependence on time $\mathcal{T}$ in formulae (4), (5) where the only really important time argument is the Solar system barycentric time t.

3   PROCEDURE OF ESTIMATION OF PULSAR PARAMETERS

We assume that all observations of the binary pulsar are of a similar quality and weight. Then one defines the timing residuals $r(t)$ as a difference between the observed number of the pulse, $\mathcal{N}^{obs}$, and the number $\mathcal{N}(t, \theta)$, predicted on the ground of our best guess to the prior unknown parameters of timing model (4), divided by the pulsar's rotational frequency ν, that is

$$r(t, \theta) = \frac{\mathcal{N}^{obs} - \mathcal{N}(t, \theta)}{\nu}, \tag{6}$$

where $\theta = \{\theta_a, a = 1, 2, ...k\}$ denotes a set of k measured parameters $[k = 14$ in the model (4)] which are shown in Table 1. It is worth noting that hereafter we use for the reason of convinuence the time argument $u = n_b t$.

If a numerical value of the parameter θ_a coincides with its true physical value $\hat{\theta}_a$, then the set of residuals would represent a physically meaningful noise $\epsilon(t)$, *i.e.*

$$r(t, \hat{\theta}) = \epsilon(t). \tag{7}$$

In practice, however, the true values of parameters are not attainable and we deal actually with their least square estimates θ_a^*. Therefore, observed residuals are fitted to the expression which is a linear function of corrections to the estimates θ_a^* of a priori unknown true values of parameters $\hat{\theta}_a$. From a Taylor expansion of the timing model in equation (4), and the fact that $r(t, \hat{\theta}) = \epsilon(t)$ one obtains

$$r(t, \theta^*) = \epsilon(t) - \sum_{a=1}^{14} \beta_a \psi_a(t, \theta^*) + O(\beta_a^2), \tag{8}$$

Table 1. List of the basic functions and parameters used in the fitting procedure. Spin parameters $\delta\mathcal{N}_0, \delta\nu, \delta\,\dot{\nu}, \delta\,\ddot{\nu}, \delta\,\dddot{\nu}$, fit rotational motion of the pulsar around its own axis. Keplerian parameters $\delta x, \delta\sigma, \delta n_b$ fit the Keplerian orbital motion of the pulsar about barycenter of the binary system. Post-Keplerian parameters $\delta\,\dot{x}, \delta\,\ddot{x}, \delta\,\dddot{x}, \delta\,\dot{n}_b, \delta\,\ddot{n}_b$ fit small observable deviations of the pulsar's orbit from the Keplerian motion caused by the effects of General Relativity, radial acceleartion, and proper motion of barycenter of the binary system with respect to the observer

Parameter	Fitting Function
$\beta_1 = \frac{\delta\mathcal{N}_0}{\nu}$	$\psi_1(t) = 1$
$\beta_2 = \frac{1}{n_b}\frac{\delta\nu}{\nu}$	$\psi_2(t) = u$
$\beta_3 = \frac{1}{2n_b^2}\frac{\delta\dot{\nu}}{\nu}$	$\psi_3(t) = u^2$
$\beta_4 = \frac{1}{6n_b^3}\frac{\delta\ddot{\nu}}{\nu}$	$\psi_4(t) = u^3$
$\beta_5 = \frac{1}{24n_b^4}\frac{\delta\dddot{\nu}}{\nu}$	$\psi_5(t) = u^4$
$\beta_6 = \frac{1}{120n_b^5}\frac{\delta\ddddot{\nu}}{\nu}$	$\psi_6(t) = u^5$
$\beta_7 = -\delta x\sin\sigma - \delta\sigma x\cos\sigma$	$\psi_7(t) = \cos u$
$\beta_8 = -\delta x\cos\sigma + \delta\sigma x\sin\sigma$	$\psi_8(t) = \sin u$
$\beta_9 = \frac{1}{n_b}\left(-\delta\,\dot{x}\cos\sigma + \delta n_b x\sin\sigma\right)$	$\psi_9(t) = u\sin u$
$\beta_{10} = \frac{1}{n_b}\left(-\delta\,\dot{x}\sin\sigma - \delta n_b x\cos\sigma\right)$	$\psi_{10}(t) = u\cos u,$
$\beta_{11} = \frac{1}{2n_b^2}\left(-\delta\,\ddot{x}\sin\sigma - \delta\,\dot{n}_b\, x\cos\sigma\right)$	$\psi_{11}(t) = u^2\cos u$
$\beta_{12} = \frac{1}{2n_b^2}\left(-\delta\,\ddot{x}\cos\sigma + \delta\,\dot{n}_b\, x\sin\sigma\right)$	$\psi_{12}(t) = u^2\sin u$
$\beta_{13} = \frac{1}{6n_b^3}\left(-\delta\,\dddot{x}\cos\sigma + \delta\,\ddot{n}_b\, x\sin\sigma\right)$	$\psi_{13}(t) = u^3\sin u$
$\beta_{14} = \frac{1}{6n_b^3}\left(-\delta\,\dddot{x}\sin\sigma - \delta\,\ddot{n}_b\, x\cos\sigma\right)$	$\psi_{14}(t) = u^3\cos u$

where the quantities $\beta_a \equiv \delta\theta_a = \theta_a^* - \hat{\theta}_a$ are the corrections to the presently unknown true values of parameters, and $\psi_a(t,\theta^*) = \left[\frac{\partial\mathcal{N}}{\partial\theta_a}\right]_{\theta=\theta^*}$ are basic fitting functions of the timing model.

In the following it is more convenient to regard the increments β_a as new parameters whose values are to be determined from the fitting procedure. The parameters β_a and fitting functions are summarized in Table 1 with asterisks omitted and time t is replaced for convenience by the function $u = n_b t$ which is the current value of orbital phase. It is worth emphasizing that the basic functions $\psi_{2a-1}, (a = 1, ..., 7)$ are even , and $\psi_{2a}, a = 1, ..., 7$ are odd. We restrict the model to 14 parameters since in practice only the first several parameters of the model are significant in fitting to the rotational and orbital phases over the available time span of observations. It is also important to understand that the smaller amount of fitting parameters one takes, the more significant the contribution of non-stationary part of low frequency noise (see discussion at the end of this section).

Now suppose that we measure m equally spaced and comparably accurate arrival times each orbit for a total of N orbital revolutions, so we have mN residuals $r_i \equiv r(t_i)$, $i = 1, ..., mN$. Standard least squares procedure (Bard 1974) gives the best fitting solution for estimates of the parameters β_a

$$\beta_a(T) = \sum_{b=1}^{14}\sum_{i=1}^{mN} L_{ab}^{-1}\psi_b(t_i)\epsilon(t_i), \qquad a = 1, ..., 14, \tag{9}$$

where the matrix of information is

$$L_{ab}(T) = \sum_{i=1}^{mN}\psi_a(t_i)\psi_b(t_i), \tag{10}$$

the matrix L_{ab}^{-1} is its inverse, and $T = NP_b$ is a total span of observational time. Matrices L_{ab} and L_{ab}^{-1} are given up to numerical factor $\frac{m}{2\pi}$ in Tables 3÷6.

Let the angular brackets denote an ensemble average over many realizations of the observational procedure. Hereafter, we assume that the ensemble average of the noise $\epsilon(t)$ is equal to zero [‡]. Hence, the mean value of any parameter β_a is equal to zero as well, *i.e.*

$$< \epsilon(t) >= 0 \qquad \longrightarrow \qquad < \beta_a >= 0. \tag{11}$$

The covariance matrix $M_{ab} \equiv < \beta_a\beta_b >$ of the parameter estimates is now given by the expression

$$M_{ab}(T) = \sum_{c=1}^{14}\sum_{d=1}^{14} L_{ac}^{-1}L_{bd}^{-1}\left[\sum_{i=1}^{mN}\sum_{j=1}^{mN}\psi_c(t_i)\psi_d(t_j)R(t_i,t_j)\right], \tag{12}$$

where $R(t_i, t_j) = < \epsilon(t_i)\epsilon(t_j) >$ is the autocovariance function of the stochastic process $\epsilon(t)$. The covariance matrix is symmetric $(M_{ab} = M_{ba})$, elements of its main diagonal give variations (or dispersions) of measured parameters $\sigma_{\beta_a} \equiv M_{aa}=< \beta_a^2 >$, and the off-diagonal terms represent the degree of statistic covariance (or correlation) between them. Covariance matrices are given explicitly in Section 6.

Subtraction of the adopted model from the observational data leads to the residuals which are dominated by the random fluctuations only. An expression for the mean-square residuals after subtracting the best-fitting solution for the estimates (9) is given by the formula

$$< r^2(T) >= \frac{1}{mN}\sum_{i=1}^{mN}\sum_{j=1}^{mN}F(t_i,t_j)R(t_i,t_j), \tag{13}$$

where the function

$$F(t_i,t_j) = \delta_{ij} - \sum_{a=1}^{14}\sum_{b=1}^{14}L_{ab}^{-1}\psi_a(t_i)\psi_b(t_j), \tag{14}$$

is called the filter function (Blandford *et al.* 1984). These expressions merely demonstrate that the amount of the background noise is reduced by the fit for the pulsar's spin and orbital parameters (Bertotti *et al.* 1983, Blandford *et al.* 1984, Bard 1974).

[‡] see Section 6 where the influence of time polynomial drift of the ensemble average of noise is treated in more detail

Thus, the observed magnitude of residuals is on the average smaller than that of the noise. This is because we have chosen the estimates θ_a^* for parameters so as to make the residuals as small as possible. Let us note that the more fitting parameters one has in the timing model, the smaller is the mean amplitude of residuals.

Another remarkable feature of (13) is that if the autocovariance function $R(t_i, t_j)$ contains products of terms of the form $\psi_c(t_i) \times f(t_j)$, where $f(t_j)$ is an arbitrary smooth function, they must disappear from the post-fit residuals (Kopeikin 1997a). The proof of the statement is based on two exact equalities:

$$\sum_{i=1}^{mN}\sum_{j=1}^{mN}\delta_{ij}\psi_c(t_i)f(t_j) = \sum_{i=1}^{mN}\psi_c(t_i)f(t_i), \tag{15}$$

and

$$\sum_{i=1}^{mN}\sum_{j=1}^{mN}\sum_{a=1}^{14}\sum_{b=1}^{14}L_{ab}^{-1}\psi_a(t_i)\psi_b(t_j)\psi_c(t_i)f(t_j) = \sum_{j=1}^{mN}\sum_{a=1}^{14}\sum_{b=1}^{14}L_{ab}^{-1}L_{ac}\psi_b(t_j)f(t_j) = \sum_{j=1}^{mN}\sum_{b=1}^{14}\delta_{bc}\psi_b(t_j)f(t_j) = \sum_{j=1}^{mN}\psi_c(t_j)f(t_j). \tag{16}$$

We have set up the hypothesis (Kopeikin 1997a) that the non-stationary part of any low-frequency noise with rational power-law spectrum can be represented as a sum of terms being products of a polynomial of time $g(t) = a_0 + a_1 t + a_2 t^2 + ...$ by a smooth function of time $f(t)$. This has been proved for a fairly general event of low-frequency noise being generated by a shot noise random process (Kopeikin 1997b) and seems to be true, if not for all, at least for a considerable number of stochastic processes. Taking large enough number of fitting spin-down parameters and corresponding fitting functions $\psi(t)$ one can represent the non-stationary component of noise as a sum of terms of the form $\psi(t_i) \times f(t_j)$. Hence, residuals for the example given do not contain the non-stationary component. In real practice, observers fit usually for the first three spin-down parameters - initial rotational phase, frequency, and frequency derivative. Such a procedure eliminates from the residuals any non-stationary noise components having spectral index $n \leq 6$.

Moreover, we emphasize that the property of the fitting procedure to filter out terms of the form of $\psi_c(t_i)f(t_j)$ does not actually depend on whether observations are equally spaced. Hence, the conclusion is that, if the autocovariance function has a non-stationary component comprised of a sum of terms of the form $\psi_c(t_i) \times f(t_j)$, one can always make the post-fit residuals depending only on the stationary part of the noise

$$< r^2(T) >= \frac{1}{mN}\sum_{i=1}^{mN}\sum_{j=1}^{mN}F(t_i, t_j)R^-(t_i, t_j) = -\frac{1}{mN}\sum_{a=1}^{14}\sum_{b=1}^{14}L_{ab}^{-1}\left[\sum_{i=1}^{mN}\sum_{j=1}^{mN}\psi_a(t_i)\psi_b(t_j)R^-(t_i, t_j)\right]. \tag{17}$$

For this reason, methods of spectral analysis in frequency domain can be applied without any restriction.

To find the asymptotic behavior of the residuals and elements of the covariance matrix as functions of time (or a number of orbital revolutions N) one needs to restrain a model of the stochastic noise process.

4   MODEL OF NOISE AND ITS AUTOCOVARIANCE FUNCTION

We have already assumed that noise consists of algebraic sum of mutually uncorrelated components of white and low-frequency noises (see Eq.(5)). White noise is due to measuring uncertainty of TOA while the low-frequency noise has terrestrial and/or astrophysical origin and becomes significant only on long enough span of observational time. Investigation of its nature is

one of the most important problems of pulsar timing since it gives us a key to much better understanding of the very deep fundamentals of physical laws governing the evolution of Nature on long time intervals. Especially important in this respect are observations of millisecond pulsars in binary systems having the best attainable accuracy and, consequently, the low level of white noise component in TOA residuals.

White noise is stationary and has a flat constant spectrum

$$S(f) = h_0, \tag{18}$$

with h_0 being a fixed parameter determined from the measured level of white noise on short time scales (see Eq. (42). The autocovariance function of white noise is

$$R(\tau) = h_0 \delta(\tau), \tag{19}$$

where $\delta(\tau)$ is the Dirac delta function and $\tau = t_i - t_j$ with t_i, t_j being moments of TOA of the i-th and j-th observations.

The generalized model of low frequency noise was developed by Kopeikin (1997b) using the shot-noise approximation. It can describe both random walk and flicker noises having, as a rule, different physical origins. The autocovariance function of the noise includes both stationary and non-stationary components which are treated on equal footing. We suppose that the autocovariance function of any stochastic process under consideration is a function of real variable and can be split algebraically into stationary and non-stationary components

$$R(t_i, t_j) = R^+(t_i, t_j) + R^-(t_i, t_j). \tag{20}$$

The function $R^+(t_i, t_j)$ describes the non-stationary part of the noise and $R^-(t_i, t_j) \equiv R^-(|t_i - t_j|)$ is its stationary counterpart. The latter can be displayed in the frequency domain through the cosine Fourier transform

$$R^-(\tau) = 2 \int_0^\infty S(f) \cos(2\pi f \tau) df, \qquad \tau \equiv t_i - t_j \tag{21}$$

where $S(f)$ is the spectrum of $R^-(\tau)$

$$S(f) = 2 \int_0^\infty R^-(\tau) \cos(2\pi f \tau) d\tau. \tag{22}$$

The stationary part of the red noise has a power-law spectrum, $S(f)$, being proportional to f^{-n} where n is called the spectral index of the noise and in the present paper $n = 1, 2, 3, 4, 5, 6$. Corresponding autocovariance functions of the stationary components are represented by either polynomial of time (for random walk noise) or polynomial of time plus logarithmic function of time multiplied by another polynomial of time (for flicker noise). Nomenclature of the corresponding names of noises, their spectra, and stationary part of the autocovariance functions are given in Table 2 which is an extract from (Kopeikin 1997b). Non-stationary parts of the autocovariance functions are expressed as a sum of terms of the form: [basic function of fitting procedure $\psi_c(t_i)$] × [a smooth function $f(t_j)$] and are given below in Section 7. As mentioned above, such terms do not contribute to post-fit timing residuals. However, they do contribute to the fitted values of the pulsar's spin parameters.

Table 2. The spectra of timing noise and the stationary part of the autocovariance functions $R^-(t_i, t_j)$ used in the paper. Constant parameters h_n, where $n = 1, 2, ..., 6$, characterize the magnitude of the noise spectrum. The quantity $\tau = t_i - t_j$, and one introduces notation $\tau_- = -\tau$, if $\tau < 0$, and $\tau_- = 0$, if $\tau \geq 0$.

Noise	Spectrum	Autocovariance Function
White Noise (WN)	h_0	$h_0 \delta(\tau)$
Flicker Noise in Phase (FPN)	$h_1 f^{-1}$	$-\frac{h_1}{\pi} \ln|\tau|$
Random Walk in Phase (RWPN)	$h_2 f^{-2}$	$-h_2 \tau_-$
Flicker Noise in Frequency (FFN)	$h_3 f^{-3}$	$\frac{1}{2\pi} h_3 \tau^2 \ln|\tau|$
Random Walk in Frequency (RWFN)	$h_4 f^{-4}$	$\frac{1}{6} h_4 \tau_-^3$
Flicker Noise in Frequency Derivative (FSN)	$h_5 f^{-5}$	$-\frac{1}{24\pi} h_5 \tau^4 \ln|\tau|$
Random Walk in Frequency Derivative (RWSN)	$h_6 f^{-6}$	$-\frac{1}{120} h_6 \tau_-^5$

Before going further, let us discuss in more detail computational aspects of our analytical approach for calculation of residuals and covariance matrices under the influence of different noise processes.

5   COMPUTATIONAL ASPECTS

In order to accomplish all calculations analytically we assume that the number of observations is so large that any sum over observing points can be approximated by the integral over the observing period T^*. For calculational convenience, we assume that observations are commenced at the moment $t_0 = -T^*/2$ and finished at $t_0 = T^*/2$ so that the interval of integration time is symmetric with respect to the origin of time scale. Such a shift of the time scale is always possible and preserves the invariance of timing formula (4). Moreover, we assume that observations are equally spaced with small enough interval of time between successive observations $\triangle t = P_b/m$ where m is the number of observations per one orbital period. A total number of orbital revolutions N is supposed to be large ($N \geq 30$).

Under the given conditions one may apply to any smooth function $f(t) \in C^\infty[-t_0, t_0]$ the Euler-Maclaurin summation formula (2.9.15) given in textbook of Davis & Rabinowitz (1984) and having the form of an asymptotic expansion

$$\sum_{i=1}^{mN} f(t_i) = \sum_{i=1}^{mN} f[-t_0 + (i-1)\triangle t] = \frac{m}{2\pi} \int_{-T}^{T} f(u)du + E(m, N), \tag{23}$$

where $T \equiv \pi N$, and the equalities $n_b \triangle t / P_b = n_b du/(2\pi)$ and $T = n_b t_0 = n_b T^*/2$ have been used[§]. Herein $E(m, N)$ is the error of approximation of the summation formula (23)

$$E(m, N) = \frac{1}{2}[f(-T) - f(T)] + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left(\frac{2\pi}{m}\right)^{2k-1} \left[f^{(2k-1)}(T) - f^{(2k-1)}(-T)\right], \tag{24}$$

where derivatives $f^{(k)}$ of the function $f(u)$ are taken with respect to the argument u, and the constants B_{2k} are called the *Bernoulli numbers* with the numerical values

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[§] New number T counts the span of observational time in terms of orbital revolutions of the binary pulsar in question. When discussing observations of a single pulsar we treat the number N as being counted in years

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42}, \qquad B_8 = -\frac{1}{30}, \qquad etc. \tag{25}$$

It is not difficult to check that the error $E(m, N)$ is decreasing, at least as $(mN)^{-1}$ comparativly to the main term, or even faster. It may be made small enough (less than 5observations per orbital period $m \geq 10$ and the number of orbital revolutions $N \geq 30$) (Wex 1997). We assume, in what follows, that this condition is fulfilled and the residual terms $E(m, N)$ are negligibly small.

Because the autocovariance function consists of a linear combination of two terms representing non-stationary and stationary components, the covariance matrix (12) is split into two liearly independent terms as well:

$$M_{ab} = M_{ab}^+ + M_{ab}^-, \tag{26}$$

where M_{ab}^+ depends only on $R^+(t_i, t_j)$ and M_{ab}^- depends only on $R^-(t_i, t_j)$. Let us underline that the shift in the origin of initial epoch of observations leads to the shift of arguments in the autocovariance functions $t_i \to t_i + T^*/2$, $t_j \to t_j + T^*/2$. It leaves the stationary part of the autocovariance function $R^-(t_i, t_j)$ invariant, depending on the difference $t_i - t_j$ only, though produces changes in the non-stationary part of $R^+(t_i, t_j)$ (see section 7 for more detail).

Calculations of integrals depending on the non-stationary part $R^+(t_i, t_j)$ of the autocovariance function are relatively easy to handle. Let us remind that the function $R^+(t_i, t_j)$ for any noise model is composed of the finite sum of products of basic functions $\psi_l(t_i)\psi_k(t_j)$, $(l, k \leq 6)$ in the event of random walk noise to which functions having the structure $\psi_l(t_i)\ln(n_b t_j + T)$, $l \leq 3$ are added in the event of flicker noise (Kopeikin 1997b). On one hand, for random walk the corresponding part of the covariance matrix will be

$$M_{ab}^+ \sim \sum_{c=1}^{14}\sum_{d=1}^{14} L_{ac}^{-1} L_{bd}^{-1} \left[\sum_{i=1}^{mN}\psi_c(t_i)\psi_l(t_i)\right] \left[\sum_{j=1}^{mN}\psi_d(t_j)\psi_k(t_j)\right]$$
$$= \left[\sum_{c=1}^{14} L_{ac}^{-1} L_{cl}\right]\left[\sum_{d=1}^{14} L_{bd}^{-1} L_{dk}\right] = \delta_{al}\delta_{bk}, \qquad (l, k \leq 6), \tag{27}$$

where δ_{ab} is the unit matrix. On the other hand, in the event of flicker noise one has additional contributions

$$M_{ab}^+ \sim \sum_{c=1}^{14}\sum_{d=1}^{14} L_{ac}^{-1} L_{bd}^{-1} \left[\sum_{i=1}^{mN}\psi_c(t_i)\psi_l(t_i)\right] \left[\sum_{j=1}^{mN}\psi_d(t_j)\ln(n_b t_j + T)\right] =$$
$$\delta_{al}\left[\frac{m}{2\pi}\sum_{d=1}^{14} L_{bd}^{-1}\int_{-T}^{T}\psi_d(u)\ln(u + T)du\right], \qquad (l \leq 3). \tag{28}$$

Relationships (27)-(28) lead to the important conclusion that, for the timing model under consideration, the non-stationary part of any autocovariance function contributes only to the elements of the covariance matrix M_{ab} with indices $a, b \leq 6$ for the random walk noise, and to the elements of $M_{ab} = M_{ba}$ with $a \leq 3, b = 1, 2, ...14$ for the flicker noise. This specific behaviour of non-stationary part of autocovariance functions was noted partially by Groth (1975) in the context of his investigation of random walk processes in pulsar timing based upon the usage of orthogonal polynomials. Our derivation of the given result is, in fact, more general as it does not depend on the particular choice of special functions used for expansion of random process and is applicable both to random walk and flicker noises as well.

Calculations of integrals depending only on the stationary part of autocovariance function are fulfilled after making a
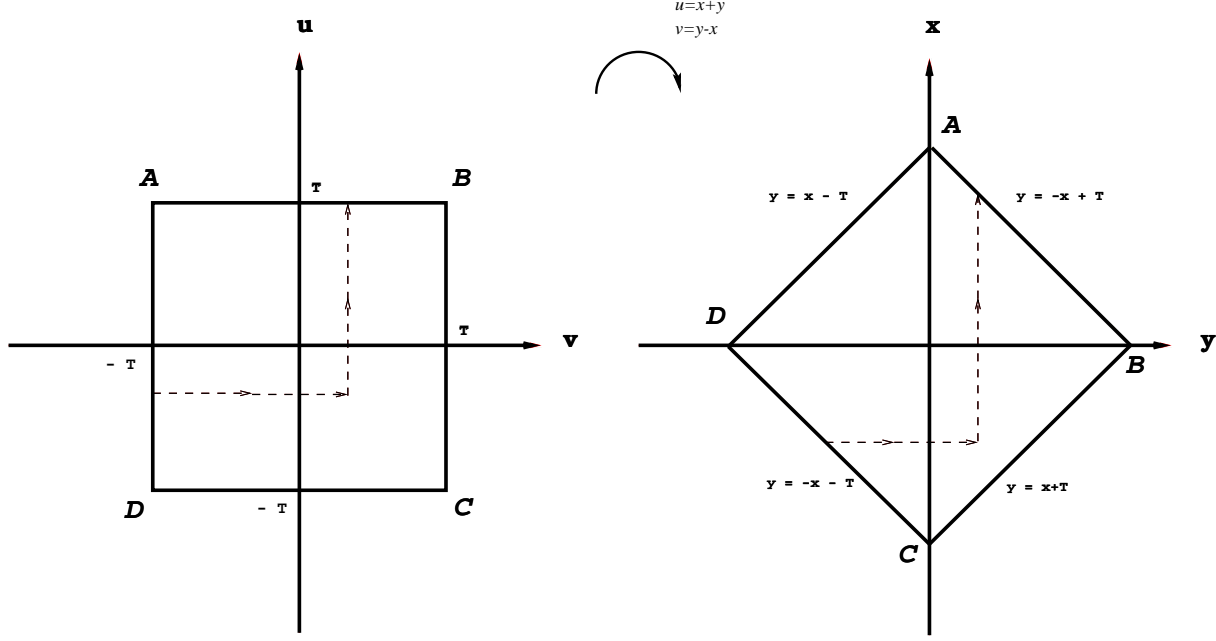
Figure 1. Geometrical illustration showing change of independent variables and transformation of the domain of integration in the phase space of two time arguments. Dashed arrowed lines indicate paths of integration.

preliminary transformation of independent variables (see Fig. 1). Let us denote $u = n_b t_i$, $v = n_b t_j$, $t_i \neq t_j$ and make a transformation

$$x = \frac{u-v}{2}, \qquad y = \frac{u+v}{2}. \tag{29}$$

It is obvious that the stationary part $R^-(t_i, t_j)$ of the autocovariance function depends only on $|x|$. Hence, the part of the covariance matrix M_{ab}^- is transformed after using formula (29) to the integral

$$M_{ab}^- = \int_{-T}^{T}\int_{-T}^{T} \psi_a(u)\psi_b(v)R^-(u-v)dudv = \int_0^T dx R^-(x)\int_0^{x-T} dy A_{ab}(x,y), \tag{30}$$

where the symmetric matrix

$$A_{ab}(x,y) \equiv -2\left[1 + (-1)^{a+b}\right]\left[\psi_a(u)\psi_b(v) + \psi_a(v)\psi_b(u)\right]. \tag{31}$$

The result shows that in the analytical approximation under consideration the elements of matrix $M_{ab}^- \neq 0$, if and only if the basic functions $\psi_a(u)$ and $\psi_b(v)$ are both either odd or even. In other words, any fitting parameter having odd number has no correlation with that having even number.

For this reason, when calculating integrals by formula (31) it is more convenient to re-group fitting parameters in order to represent the matrix $A_{ab}(x,y)$ into a block-diagonal form

$$A_{ab} \qquad \Rightarrow \qquad \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{32}$$

where the left upper block $A_1 = A_{2a-1,2b-1}$ depends only on odd fitting functions, and the right lower block $A_2 = A_{2a,2b}$ depends only on even ones $(a,b = 1,...,7$. Due to this re-arrangement of parameters, the matrix L_{ab} and consequently the inverse matrix L_{ab}^{-1} are also reduced to the block-diagonal form having the same structure as in (32). The advantage of this re-arrangement is that one can work with each of the block matrices independently, simplifying the calculations considerably.

Nevertheless, it is worthwhile to remind that such simplification can be achieved in the main approximation only. If one took terms of higher order in the expansion (23) the re-arrangement of elements of matrix M_{ab}^{-} would be worthless since odd and even fitting parameters get correlated. In such event one can make a progress in calculations only if numerical methods are implied. This is the case one meets in a real practice.

In the analytical approximation under consideration the integral (30) was calulated using the enhanced version 2.2.3 of MATHEMATICA for Microsoft Windows. It turns out that the outcome of calculation can be always expressed either as a polynomial and trigonometric functions of time in the event of random walk, or as a polynomial, trigenometric functions, and sine/cosine-integrals in the event of flicker noise. Since the number of observations mN has been assumed to be very large all functions appearing in elements of covariance matrix (30) have been expanded into asymptotic series with respect to the small parameter $\varepsilon = \frac{1}{\pi N} \equiv \frac{1}{T}$. Then, only the first term of the expansions has been retained and all residual terms have been abandoned. An explicit result of such calcualtion is, for instance, the matrix of information $L_{ab} = \frac{m}{2\pi} C_{ab}$ given in Tables 3 and 4. Analytical expressions for residuals and elements of covariance matrices of pulsar's fitting parameters in presence of white and red noises are given in Section 7.

6   TREATMENT OF POLYNOMIAL DRIFT OF TIMING NOISE

For mathematical convenience, in this section we shall assume that the number of spin-down parameters of timing model is equal to M and the total number of fitting parameters is K [¶]. Up to now, we have been concerned mainly with the mathematical formulation of procedure of estimation of pulsar's parameters under the assumption that noise has only random fluctuations, that is equation (11) is true. However, nonrandom variations (drifts) of the noise do exist and can be sometimes modeled by deterministic polynomial functions of time as it takes place in description of electromagnetic breaking of pulsar's rotation. Hence, it is conceivable to imagine that the ensemble average of noise is modeled by a $(P-1)$-th degree polynomial

$$< \epsilon(t) > = \sum_{k=1}^{P} \alpha_k t^{k-1} = \sum_{k=1}^{M} n_b^{-k+1} \alpha_k \psi_k(t) + \left(\alpha_{M+1} t_i^M + ... + \alpha_P t_i^{P-1} \right), \tag{33}$$

where α_k are constants which numerical values get smaller as k increases. From the definition of fitting parameters and fitting functions as well as equation (9) it is easy to verify that the mean value of the parameters reads as

$$< \beta_a(T) > = \sum_{b=1}^{K} \sum_{k=1}^{P} \sum_{i=1}^{mN} L_{ab}^{-1} \alpha_k \psi_b(t_i) t_i^{k-1} = \sum_{b=1}^{K} \sum_{k=1}^{M} \sum_{i=1}^{mN} L_{ab}^{-1} n_b^{-k+1} \alpha_k \psi_b(t_i) \psi_k(t_i) + \sum_{b=1}^{K} \sum_{k=M+1}^{P} \sum_{i=1}^{mN} L_{ab}^{-1} \alpha_k \psi_b(t_i) t_i^{k-1}$$

$$= \sum_{k=1}^{M} n_b^{-k+1} \delta_{ak} \alpha_k + \sum_{b=1}^{K} \sum_{i=1}^{mN} L_{ab}^{-1} \psi_b(t_i) \left(\alpha_{M+1} t_i^M + ... + \alpha_P t_i^{P-1} \right). \tag{34}$$

Thus, we conclude that until number P of the polynomial's coefficients (33) is less or equal than number M of spin-down parameters being used in deterministic timing model it changes only the mean values of the first M parameters. In such event one can make several more iterations in the procedure of minimization of square of residuals in order to eliminate completely

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[¶] In particular, the numbers M and K are equal to $M = 6$ and $K = 14$ respectively in the timing model described by equation (4)

the polynomial drift from the ensemble average of the noise by means of re-shifting first P spin-down parameters. As a result, one will get $< r(t) >= 0$.

If $P > M$, then

$$< r(t) >= \alpha_{M+1}t^M + ... + \alpha_P t^{P-1} - \sum_{a=1}^{K}\sum_{b=1}^{K}L_{ab}^{-1}\psi_a(t)\left[\sum_{i=1}^{mN}\psi_b(t_i)(\alpha_{M+1}t_i^M + ... + \alpha_P t_i^{P-1})\right]. \tag{35}$$

One recognizes that in this event $< r(t) >\neq 0$ though in practice it can be already within the limit of rounding error of numerical computations. It is worth emphasizing that even if it is the case, some fitted parameters β_a can, nevertheless, have the mean value lying above the limit of the rounding error. We conclude that it is impossible to make the mean value for the residuals and all fitting parameters equals to zero if the number of spin-down parameters in timing model is less than number of coefficients in polynomial drift of the ensemble average of noise. This conclusion holds true for both white and red noise regimes.

Let us turn attention now to the calculation of residuals in the presence of polynomial drift of the noise's ensemble average (33). The autocovariance function of noise reads now as

$$< \epsilon(t_i)\epsilon(t_j) >= R(t_i, t_j) + \sum_{k=1}^{P}\sum_{l=1}^{P}\alpha_k\alpha_l t_i^{k-1}t_j^{l-1}, \tag{36}$$

where $R(t_i, t_j)$ is the autocovariance function of the noise with zero-valued ensemble average. Influence of the function $R(t_i, t_j)$ on residuals and covariance matrix of fitting parameters is explicitly demonstrated in the next section. Calculation of ensemble average of residuals with making use of expression (36) and the property that fitting procedure filters out all terms of the form $\psi_a(t_i)f(t_j)$ yields

$$< r^2(T) >= \frac{1}{mN}\sum_{i=1}^{mN}\sum_{j=1}^{mN}F(t_i, t_j)\left[R^-(t_i, t_j) + \sum_{k=M+1}^{P}\sum_{l=M+1}^{P}\alpha_k\alpha_l t_i^{k-1}t_j^{l-1}\right]. \tag{37}$$

Thus, one can see that if the condition $P < M + 1$ holds, then the polynomial drift of noise does not contribute to the ensemble mean value of square of timing residuals. In the opposite situation, when $P \geq M + 1$, there extra contribution exist and there is a danger of confusion of white noise having a polynomial drift with a red noise without such a drift.

Regarding covariance matrix of parameters in presence of polynomial drfit of noise it is not so difficult to generalization (12) using expression (36) for the autocovariance function of noise. It results in

$$M_{ab} = M_{ab}^- + M_{ab}^+ + \Delta M_{ab}, \tag{38}$$

where the stationary part, M_{ab}^-, the non-stationary part, M_{ab}^+, and the increment ΔM_{ab} of covariance matrix read, correspondingly, as

$$M_{pq}^-(T) = \sum_{c=1}^{K}\sum_{d=1}^{K}L_{pc}^{-1}L_{qd}^{-1}\left[\sum_{i=1}^{mN}\sum_{j=1}^{mN}\psi_c(t_i)\psi_d(t_j)R^-(t_i, t_j)\right], \tag{39}$$

$$M_{pq}^+(T) = \sum_{c=1}^{K}\sum_{d=1}^{K}L_{pc}^{-1}L_{qd}^{-1}\sum_{i=1}^{mN}\sum_{j=1}^{mN}\psi_c(t_i)\psi_d(t_j)R^+(t_i, t_j) + \sum_{k=1}^{M}\sum_{l=1}^{M}n_b^{-k-l+2}\delta_{pk}\delta_{ql}\alpha_k\alpha_l$$

$$+2\left[\sum_{k=1}^{M}n_b^{-k+1}\delta_{pk}\alpha_k\right]\left[\sum_{c=1}^{K}\sum_{j=1}^{mN}L_{qd}^{-1}\psi_d(t_j)\sum_{l=M+1}^{P}\alpha_l t_j^{l-1}\right], \tag{40}$$

$$\Delta M_{ab}(T) = \sum_{c=1}^{K}\sum_{d=1}^{K} L_{ac}^{-1} L_{bd}^{-1} \left[\sum_{i=1}^{mN}\sum_{j=1}^{mN} \psi_c(t_i)\psi_d(t_j) \sum_{k=M+1}^{P}\sum_{l=M+1}^{P} \alpha_k\alpha_l t_i^{k-1} t_j^{l-1} \right]. \qquad (41)$$

Herein, we have assumed that the non-stationary part of autocovariance function of noise $R^+(t_i, t_j)$ consists only of products $\psi_a(t_i)f(t_j)$ with $a \le M$.

Analayzing the structure of relationships (38)-(41) we draw the following conclusions: 1) if the number of coefficients P in the polynomial drift of noise is less than or equal to the number M of spin-down parameters in the timing model, then it changes only non-stationary part M_{ab}^+ of the covariance matrix for spin-down parameters; -2) if the number of coefficients P in the polynomial drift of noise is more than or equal to the number M of spin-down parameters in the timing model, then it influences the non-stationary part of the covariance matrix for spin-down and orbital parameters and gives rise to ΔM_{ab}. Hence, if we do not account for the polynomial drift properly it may worsen numerical boundaries for true estimates of variances and covariances of fitting parameters.

At this point, it is worthwhile to emphasize that in order to make effects of polynomial drift of noise as small as possible one has to introduce to the timing model as many spin-down parameters as it is required by observational accuracy and optimization of the least-square minimization procedure.

7 RESIDUALS, COVARIANCE MATRIX, AND VARIANCES OF MEASURED PARAMETERS

7.1 General Comments

In what follows we assume that effect of fitting spin-down parameters is so suppressive that the polynomial drift of the noise is completely eliminated from all residuals and mean values of parameters. When calculating covariance matrices of parameters, it is worth keeping in mind that if one takes only stationary components of low frequency noise it is not sufficient to demand the variances of all fitting parameters to be positive since the first several spin-down parameters can have variances with negative values (see Tables (9)-(16) for more detail). Negative variances are, of course, physically meaningless. Hence, we conclude that it is incorrect to disregard the non-stationary part of red noise for it hampers the physical interpretation of these variances. It turns out that, in order to obtain positive values for variances of all fitted parameters, one must account for the contribution from the non-stationary component of noise as well. This clearly demonstrates the role of the non-stationary noise component in the fitting procedure. The energy distribution amongst the fitted parameters is such that the non-stationary part of red noise contains as large amount of energy as the stationary part does (or even more) for the first several parameters.

An additional point to note is that, if the number of spin-down parameters is not big enough, the non-stationary part of the autocovariance function can not be represented as a linear combination of products of a smooth function $f(t)$ by the fitting function $\psi_a(t)$, and complete filtering out non-stationary component of noise from residuals is impossible. In such a case, if only the stationary part of the noise is accounted for and the non-stationary part is omitted, a negative mean value of residuals can be obtained that is physically unadmissible. Taking into account the non-stationary noise component brings the mean value of residuals back to the positive value. By this consideration, we would like to underline the role which non-stationary part of noise plays in fitting procedure. We consider those models of low frequency noise which take into

Table 3. Elements of matrix of information C_{ab}. Quantities $\mathrm{T} = \pi N$ and $Q = \cos \mathrm{T}$.

	1	3	5	7	9	11	13
1	$2\mathrm{T}$	.	.	.	.	.	.
3	$\frac{2\mathrm{T}^3}{3}$	$\frac{2\mathrm{T}^5}{5}$	.	.	.	.	.
5	$\frac{2\mathrm{T}^5}{5}$	$\frac{2\mathrm{T}^7}{7}$	$\frac{2\mathrm{T}^9}{9}$	.	.	.	.
7	0	$4\mathrm{T}Q$	$8\mathrm{T}^3 Q$	T	.	.	.
9	$-2\mathrm{T}Q$	$-2\mathrm{T}^3 Q$	$-2\mathrm{T}^5 Q$	$-\frac{\mathrm{T}}{2}$	$\frac{\mathrm{T}^3}{2}$	.	.
11	$4\mathrm{T}Q$	$8\mathrm{T}^3 Q$	$12\mathrm{T}^5 Q$	$\frac{\mathrm{T}^3}{3}$	$-\frac{\mathrm{T}^3}{2}$	$\frac{\mathrm{T}^5}{5}$	.
13	$-2\mathrm{T}^3 Q$	$-2\mathrm{T}^5 Q$	$-2\mathrm{T}^7 Q$	$-\frac{\mathrm{T}^3}{2}$	$\frac{\mathrm{T}^5}{5}$	$-\frac{\mathrm{T}^5}{2}$	$\frac{\mathrm{T}^7}{7}$

account only stationary component to be incomplete and can lead to erroneous results. For this reason, we are very careful in dealing with the non-stationary noise component in order to properly take into account or to prove that it is insignificant for calculations. In this context, let us note once again that we have made a shift of the time scale which changes the orbital phase $u \mapsto u + \pi \mathrm{T}$ and introduces additional terms in the non-stationary part of the autocovariance function in comparison with expressions for $R^+(t_i, t_j)$ having been given in Kopeikin (1997b).

With these comments we are ready to demonstrate how white and red noise affect pulsar timing residuals and covariance matrices of fitting parameters.

7.2   White Phase Noise

The spectral power of white noise is constant. For this reason, the timing residuals are constant as well

$$< r^2 > = h_0, \tag{42}$$

and the covariance matrix of measured parameters M_{ab} coincides with the inverse matrix of information (10) L_{ab}^{-1}. The matrix of information $L_{ab} = \frac{m}{2\pi} C_{ab}$, and the elements of the inverse matrix C_{ab}^{-1} are given in Tables 3 - 6.

On multiplying the main diagonal of L_{ab}^{-1} by the magnitude of the measurement error h_0 yields the variances of the measured parameters. One can see that precision of the parameter determination depends both on the rate of making observations m and on the number of orbital revolutions N during the observational session. This is a distinctive feature of timing measurements in the event when white noise is the only source of errors. We also note that as the observational span T increses the variances of all parameters decrease. Thus, until white noise dominates in observed timing residuals, the accuracy of determination of all parameters will be improved. It is worth emphasizing as well that there is a similarity in the precision of determination of spin and orbital parameters. Indeed, if one compares variances for the rotational phase and its time derivatives with corresponding quantities for orbital motion one finds that dependence on T is essentially the same. For making this statement more obvious we give relationships between orbital parameters and parameters of the timing model shown in Table 1

$$< (\delta\sigma)^2 > \quad = \quad \frac{1}{x^2}\left(< \beta_7^2 > \cos^2 \sigma + < \beta_8^2 > \sin^2 \sigma\right) , \tag{43}$$

Table 4. Elements of matrix of information C_{ab}. Quantities $T = \pi N$ and $Q = \cos T$.

	2	4	6	8	10	12	14
2	$\frac{2T^3}{3}$	.	.	.	.	.	.
4	$\frac{2T^5}{5}$	$\frac{2T^7}{7}$	.	.	.	.	.
6	$\frac{2T^7}{7}$	$\frac{2T^9}{9}$	$\frac{2T^{11}}{11}$	.	.	.	.
8	$-2TQ$	$-2T^3Q$	$-2T^5Q$	T	.	.	.
10	$4TQ$	$8T^3Q$	$12T^5Q$	$-\frac{T}{2}$	$\frac{T^3}{3}$	.	.
12	$-2T^3Q$	$-2T^5Q$	$-2T^7Q$	$\frac{T^3}{3}$	$-\frac{T^3}{2}$	$\frac{T^5}{5}$	.
14	$8T^3Q$	$12T^5Q$	$16T^7Q$	$-\frac{T^3}{2}$	$\frac{T^5}{5}$	$-\frac{T^5}{2}$	$\frac{T^7}{7}$

Table 5. Elements of the covariance matrix C_{ab}^{-1} of pulsar's parameters for white noise in phase. Quantities $T = \pi N$ and $Q = \cos T$.

	1	3	5	7	9	11	13
1	$\frac{225}{128T}$	.	.	.	.	.	.
3	$-\frac{525}{64T^3}$	$\frac{2205}{32T^5}$	.	.	.	.	.
5	$\frac{945}{128T^5}$	$-\frac{4725}{64T^7}$	$\frac{11025}{128T^9}$	.	.	.	.
7	$\frac{315Q}{8T^3}$	$-\frac{945Q}{2T^5}$	$\frac{4725Q}{8T^7}$	$\frac{9}{4T}$	.	.	.
9	$-\frac{225Q}{16T^3}$	$\frac{1575Q}{8T^5}$	$-\frac{4725Q}{16T^7}$	$\frac{45}{8T^3}$	$\frac{75}{4T^3}$	.	.
11	$-\frac{675Q}{4T^5}$	$\frac{7875Q}{4T^7}$	$-\frac{4725Q}{2T^9}$	$-\frac{15}{4T^3}$	$-\frac{225}{8T^5}$	$\frac{45}{4T^5}$	.
13	$\frac{525Q}{16T^5}$	$-\frac{3675Q}{8T^7}$	$\frac{11025Q}{16T^9}$	$-\frac{105}{8T^5}$	$-\frac{105}{4T^5}$	$\frac{525}{8T^7}$	$\frac{175}{4T^7}$

Table 6. Elements of the covariance matrix C_{ab}^{-1} of pulsar's parameters for white noise in phase. Quantities $T = \pi N$ and $Q = \cos T$.

	2	4	6	8	10	12	14
2	$\frac{3675}{128T^3}$	.	.	.	.	.	.
4	$-\frac{6615}{64T^5}$	$\frac{14175}{32T^7}$	.	.	.	.	.
6	$\frac{10395}{128T^7}$	$-\frac{24255}{64T^9}$	$\frac{43659}{128T^{11}}$	.	.	.	.
8	$-\frac{315Q}{16T^3}$	$\frac{945Q}{8T^5}$	$-\frac{2079Q}{16T^7}$	$\frac{9}{4T}$	.	.	.
10	$\frac{4725Q}{2T^5}$	$-\frac{51975Q}{4T^7}$	$\frac{51975Q}{4T^9}$	$\frac{45}{8T^3}$	$\frac{75}{4T^3}$	.	.
12	$\frac{1575Q}{16T^5}$	$-\frac{4725Q}{8T^7}$	$\frac{10395Q}{16T^9}$	$-\frac{15}{4T^3}$	$-\frac{225}{8T^5}$	$\frac{45}{4T^5}$	.
14	$-\frac{40425Q}{8T^7}$	$\frac{55125Q}{2T^9}$	$-\frac{218295Q}{8T^{11}}$	$-\frac{105}{8T^5}$	$-\frac{105}{4T^5}$	$\frac{525}{8T^7}$	$\frac{175}{4T^7}$

$$\frac{<(\delta n_b)^2>}{n_b^2} = \frac{1}{x^2}\left(<\beta_{10}^2>\cos^2\sigma+<\beta_9^2>\sin^2\sigma\right) \; , \tag{44}$$

$$\frac{<(\delta\dot{n}_b)^2>}{n_b^2} = \frac{4n_b^2}{x^2}\left(<\beta_{11}^2>\cos^2\sigma+<\beta_{12}^2>\sin^2\sigma\right) \; , \tag{45}$$

$$\frac{<(\delta\ddot{n}_b)^2>}{n_b^2} = \frac{36n_b^4}{x^2}\left(<\beta_{14}^2>\cos^2\sigma+<\beta_{13}^2>\sin^2\sigma\right) \; , \tag{46}$$

$$<(\delta x)^2> = \left(<\beta_8^2>\cos^2\sigma+<\beta_7^2>\sin^2\sigma\right) \; , \tag{47}$$

$$<(\delta\dot{x})^2> = n_b^2\left(<\beta_9^2>\cos^2\sigma+<\beta_{10}^2>\sin^2\sigma\right) \; , \tag{48}$$

$$<(\delta\ddot{x})^2> = 4n_b^4\left(<\beta_{12}^2>\cos^2\sigma+<\beta_{11}^2>\sin^2\sigma\right) \; , \tag{49}$$

$$<(\delta\dddot{x})^2> = 36n_b^6\left(<\beta_{13}^2>\cos^2\sigma+<\beta_{14}^2>\sin^2\sigma\right) \; . \tag{50}$$

Calculation of variances of orbital parameters is accomplished using results shown in Tables 3-6. It is remarkable that there is a symmetry between variances of odd and even parameters of the timing model, so that in case of white noise dominance one has, for example, $<\beta_7^2>=<\beta_8^2>$, and so on. Because of this symmetry there is no dependence of variances of orbital parameters on the initial orbital phase σ. From equations (43)-(50) we have

$$<(\delta\sigma)^2> = \frac{9h_0}{4x^2\mathrm{T}} \; , \tag{51}$$

$$\frac{<(\delta n_b)^2>}{n_b^2} = \frac{75h_0}{4x^2\mathrm{T}^3} \; , \tag{52}$$

$$\frac{<(\delta\dot{n}_b)^2>}{n_b^2} = \frac{45h_0n_b^2}{x^2\mathrm{T}^5} \; , \tag{53}$$

$$\frac{<(\delta\ddot{n}_b)^2>}{n_b^2} = \frac{1575h_0n_b^4}{x^2\mathrm{T}^7} \; , \tag{54}$$

$$<(\delta x)^2> = \frac{9h_0}{4\mathrm{T}} \; , \tag{55}$$

$$<(\delta\dot{x})^2> = \frac{75h_0n_b^2}{4\mathrm{T}^3} \; , \tag{56}$$

$$< (\delta\ddot{x})^2 > \quad = \quad \frac{45 h_0 n_b^4}{\mathrm{T}^5} \, , \tag{57}$$

$$< (\delta\dddot{x})^2 > \quad = \quad \frac{1575 h_0 n_b^6}{\mathrm{T}^7} \, , \tag{58}$$

which can be easily compared with variances of corresponding spin-down parameters in Tables 5, 6. In particular, we find that ratio of root squares of variances of the spin-down parameters to corresponding parameters describing time evolution of orbital phase is approximately equal to x/P_b, that is $< \delta\mathcal{N}_0/\nu > / < \delta\mathcal{T}_0 > \sim x/P_b$, $< \delta\nu/\nu > / < \delta P_b/P_b > \sim x/P_b$, $< \delta\dot{\nu}/\nu > / < \delta\dot{P}_b/P_b > \sim x/P_b$, *etc.* As a concequence of this observation we conclude that in the case of domination of white noise spin-down parameters of binary pulsars are always determined better than corresponding parameters of orbital phase.

7.3 Flicker Noise in Phase ($1/f$ noise)

Pulsar timing residuals grow with time according to the relationship

$$< r^2 > = h_1 \left[\ln(2\mathrm{T}) - \frac{101}{30} \right] , \tag{59}$$

where h_1 is a constant number characterizing intensity of the noise.

The non-stationary part of autocovariance function for PN random walk is represented as (Kopeikin 1997b):

$$R^+(t_i, t_j) = h_1 \left[\ln(u + \mathrm{T}) + \ln(v + \mathrm{T}) \right] , \tag{60}$$

and can be expressed through the basic functions

$$R^+(t_i, t_j) = h_1 \left[\psi_1(v) \ln(u + \mathrm{T}) + \psi_2(u) \ln(v + \mathrm{T}) \right] . \tag{61}$$

From equation (28), the contribution from the non-stationary part of the flicker noise to the covariance matrix is given by the integral

$$M_{ab}^+ = h_1 \left[\delta_{1a} \sum_{c=1}^{14} L_{bc}^{-1} \int_{-\mathrm{T}}^{\mathrm{T}} \psi_c(u) \ln(u + \mathrm{T}) du + \delta_{1b} \sum_{c=1}^{14} L_{ac}^{-1} \int_{-\mathrm{T}}^{\mathrm{T}} \psi_c(u) \ln(u + \mathrm{T}) du \right] . \tag{62}$$

It is clear from (62) that the only non-zero elements of the matrix M_{ab}^+ can be $M_{1b}^+ = M_{b1}^+$ ($b = 1, ..., 14$). The most interesting component is the contribution of the non-stationary part to the variance of the first measured parameter

$$M_{11}^+ = -\frac{361}{240} + 2\ln(2\mathrm{T}) \tag{63}$$

The matrix M_{ab}^- is generated by the stationary part of the autocovariance function and is given in Tables 5 and 8. Let us note that, as expected, variances of all measured parameters including the first one are positive quantities.

Comparing results of the variance calculations with those obtained in previous section for white noise we note that flicker noise in phase worsen our ability in determining variances of spin-down parameters. At the same time flicker noise does not disturb variances of orbital parameters and symmetry between pairs of odd and even parameters. For this reason, using general formulae (43)-(50) one can see that variances of the orbital parameters do not depend on the initial orbital phase σ and have the same dependence on the total span of observation T as it was in case of white noise. We would like to emphasize the appearance of logarithmic terms in timing residuals and the covariance matrices given in Tables 5 and 8. Logarithmic

Table 7. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for flicker noise in phase. Quantities $T = \pi N$, $Q = \cos T$, and $\Gamma = \gamma + \ln(2T)$. The magnitude of the noise h_1 is omitted.

		1	3	5	7	9	11	13
	1	$\frac{7473}{2560} - \ln(2T)$	.	.	.	.	.	.
	3	$-\frac{1631}{256T^2}$	$\frac{5145}{128T^4}$	.	.	.	.	.
a	5	$\frac{2289}{512T^4}$	$-\frac{9555}{256T^6}$	$\frac{19845}{512T^8}$	.	.	.	.
	7	$\frac{3Q}{64T^2}[119+60\Gamma]$	$-\frac{315Q}{32T^4}[3+4\Gamma]$	$-\frac{315Q}{64T^4}[1-12\Gamma]$	$\frac{9\pi}{4T}$	.	.	.
	9	$-\frac{4Q}{16T^2}$	$\frac{735Q}{16T^4}$	$-\frac{945Q}{16T^6}$	$\frac{75\pi}{4T^3}$	$\frac{75\pi}{4T^3}$	.	.
	11	$-\frac{Q}{64T^4}[1273+900\Gamma)]$	$-\frac{105Q}{32T^6}[17+60\Gamma]$	$\frac{315Q}{64T^8}[29-60\Gamma]$	$-\frac{15\pi}{4T^3}$	$-\frac{315\pi}{4T^5}$	$\frac{45\pi}{4T^5}$	.
	13	$\frac{28Q}{3T^4}$	$-\frac{1715Q}{16T^6}$	$\frac{2205Q}{16T^8}$	$-\frac{105\pi}{4T^5}$	$-\frac{105\pi}{4T^5}$	$\frac{525\pi}{4T^7}$	$\frac{175\pi}{4T^7}$

[a] Hereafter the letter $\gamma = 0.577215...$ denotes the Euler's constant.

Table 8. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for flicker noise in phase. Quantities $T = \pi N$, $Q = \cos T$, and $\Gamma = \gamma + \ln(2T)$. The magnitude of noise h_1 is omitted.

	2	4	6	8	10	12	14
2	$\frac{4851}{256T^2}$	.	.	.	.	.	.
4	$-\frac{3381}{64T^4}$	$\frac{12495}{64T^6}$	.	.	.	.	.
6	$\frac{46893}{1280T^6}$	$-\frac{4851}{32T^8}$	$\frac{160083}{1280T^{10}}$	.	.	.	.
8	$-\frac{693Q}{160T^2}$	$\frac{357Q}{16T^4}$	$-\frac{693Q}{32T^6}$	$\frac{9\pi}{4T}$	.	.	.
10	$\frac{315Q}{64T^4}[61+20\Gamma]$	$-\frac{105Q}{32T^6}[341+180\Gamma]$	$\frac{2079Q}{64T^8}[21+20\Gamma]$	$-\frac{15\pi}{2T^3}$	$\frac{75\pi}{4T^3}$	.	.
12	$\frac{693Q}{32T^4}$	$-\frac{1785Q}{16T^6}$	$\frac{3465Q}{32T^8}$	$-\frac{15\pi}{4T^3}$	$\frac{45\pi}{2T^5}$	$\frac{45\pi}{4T^5}$	.
14	$-\frac{147Q}{64T^6}[261+100\Gamma]$	$\frac{735Q}{32T^8}[91+60\Gamma]$	$-\frac{1617Q}{64T^{10}}[43+60\Gamma]$	$-\frac{21}{2T^6}[289-120\Gamma]$	$-\frac{105\pi}{4T^5}$	$\frac{315}{T^8}[49-20\Gamma]$	$\frac{175\pi}{4T^7}$

terms are characteristic for any flicker noise as it will be shown in subsequent sections. However, practical observation of the logarithmic terms in temporal behaviour of residuals and variances of measured parameters is a challenge for observers since it is difficult to distinguish $\ln T$ from constant if observational span is not long enough. Perhaps, this explains why logarithmic behavior of timing residuals and variances have not been yet found and flicker noise with spectral index $s = 1$ has not been ever identified in pulsar timing observations.

7.4   Random Walk in Phase ($1/f^2$ noise)

The timing residuals grow proportionally to the number of orbital revolutions N

$$< r^2 > = \frac{6}{143} h_2 T, \tag{64}$$

where h_2 is a constant number characterizing intensity of the noise. The non-stationary part of autocovariance function for PN random walk is represented as (Kopeikin 1997b):

Table 9. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for random walk in phase. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_2 is omitted.

	1	3	5	7	9	11	13
1	$-\frac{45T}{704}$	.	.	.	.	.	.
3	$-\frac{135}{176T}$	$\frac{525}{176T^3}$	.	.	.	.	.
5	$\frac{245}{704T^3}$	$-\frac{105}{44T^5}$	$\frac{1575}{704T^7}$	.	.	.	.
7	$\frac{189Q}{176T}$	$-\frac{945Q}{88T^3}$	$-\frac{2205Q}{176T^5}$	$\frac{9}{4T}$	.	.	.
9	$-\frac{45Q}{176T}$	$\frac{225Q}{88T^3}$	$-\frac{525Q}{176T^5}$	$\frac{15\pi}{88T^3}$	$\frac{1275}{44T^3}$	.	.
11	$-\frac{855Q}{176T^3}$	$\frac{4275Q}{88T^5}$	$-\frac{9975Q}{176T^7}$	$-\frac{15}{4T^3}$	$\frac{765}{88T^5}$	$\frac{45}{4T^5}$	.
13	$\frac{105Q}{176T^3}$	$-\frac{525Q}{88T^5}$	$\frac{1225Q}{176T^7}$	$\frac{3045}{88T^5}$	$-\frac{2205}{44T^5}$	$-\frac{11025}{88T^7}$	$\frac{4375}{44T^7}$

Table 10. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for random walk in phase. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_2 is omitted.

	2	4	6	8	10	12	14
2	$\frac{1575}{832T}$	.	.	.	.	.	.
4	$-\frac{175}{52T^3}$	$\frac{2205}{208T^5}$	.	.	.	.	.
6	$\frac{1701}{832T^5}$	$-\frac{1575}{208T^7}$	$\frac{4851}{832T^9}$	.	.	.	.
8	$-\frac{45Q}{208T}$	$\frac{105Q}{104T^3}$	$-\frac{189Q}{208T^5}$	$\frac{135}{52T}$	.	.	.
10	$\frac{7425Q}{208T^3}$	$-\frac{17325Q}{104T^5}$	$\frac{31185Q}{208T^7}$	$-\frac{6915}{104T^3}$	$\frac{75}{4T^3}$	.	.
12	$\frac{225Q}{208T^3}$	$-\frac{525Q}{104T^5}$	$\frac{945Q}{208T^7}$	$-\frac{285}{52T^3}$	$\frac{31455}{104T^5}$	$\frac{1035}{52T^5}$	.
14	$-\frac{16275Q}{208T^5}$	$\frac{37975Q}{104T^7}$	$-\frac{68355Q}{208T^9}$	$\frac{11655}{104T^5}$	$-\frac{105}{4T^5}$	$-\frac{58275}{104T^7}$	$\frac{175}{4T^7}$

$$R^+(t_i, t_j) = h_2 T \left\{ \psi_1(t_i)\psi_1(t_j) + \frac{1}{2T} \left[\psi_1(t_i)\psi_2(t_j) + \psi_2(t_i)\psi_1(t_j) \right] \right\}. \tag{65}$$

Using Eq. (27) the covariance matrix for non-stationary part of PN results in

$$M_{ab}^+ = h_2 T \left[\delta_{a1}\delta_{b1} + \frac{1}{2T}\delta_{a1}\delta_{b2} + \frac{1}{2T}\delta_{b1}\delta_{a2} \right]. \tag{66}$$

Stationary part of the PN noise gives the covariance matrix displayed in Tables 9 and 10. One can see that the random walk in phase makes it impossible to track the rotational phase of the pulsar after amount of orbital revolutions exeeds $(2816\pi)/(659\nu^2 h_2) \simeq 13.4/(\nu^2 h_2)$ since the moment when random walk in phase becomes the dominant source of noise in timing residuals. In this case the parameter $\mathcal{N}_0$ becomes non-informative[‖]. All other spin-down parameters can be measured but accuracy of their measurement is lower comparatively to the cases of white and/or flicker noises in phase. On the other

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[‖] A parameter is called informative if its mean numerical value obtained in the fitting procedure is much less than its variance. In the opposite case, the parameter is called non-informative. We also call the initial rotational and orbital phases, $\mathcal{N}_0$ and σ, informative parameters if their variances do not exeed 2π. From a physical point of view, some of the fitting parameters become non-informative

hand, time dependence of variances of orbital parameters is the same as it was in the case of white noise. Hence, we are still able to determine the initial orbital phase σ, orbital frequency n_b, and so on. However, the symmetry between pairs of corresponding odd and even parameters of the timing model under consideration is not preserved. For this reason, accuracy in determination of variances of orbital parameters slightly depends on the initial orbital phase (which remains to be the informative parameter).

7.5 Flicker Noise in Frequency ($1/f^3$ noise)

The timing residuals are proportional to the square of the number of orbital revolutions

$$< r^2 > = \frac{1}{105} h_3 \mathrm{T}^2, \tag{67}$$

where h_3 is a constant number characterizing intensity of the noise. The non-stationary part of autocovariance function for flicker noise in frequency is represented as (Kopeikin 1997b):

$$
\begin{aligned}
R^+(t_i, t_j) \quad &= \tfrac{1}{2} h_3 \mathrm{T}^2 \Bigg\{ -\psi_1(v)\psi_1(u) - \tfrac{1}{\mathrm{T}}\psi_1(v)\psi_2(u) + -\tfrac{1}{\mathrm{T}}\psi_2(v)\psi_1(u) - \tfrac{1}{\mathrm{T}^2}\psi_2(v)\psi_2(u) + \\[2mm]
&\quad \left[\psi_1(v) + \tfrac{2}{\mathrm{T}}\psi_2(v) + \tfrac{2}{\mathrm{T}}\psi_1(v)u - \tfrac{1}{\mathrm{T}^2}\psi_1(v)u^2 \right] \ln(u + \mathrm{T}) + \\[2mm]
&\quad + \left[\psi_1(u) + \tfrac{2}{\mathrm{T}}\psi_2(u) + \tfrac{2}{\mathrm{T}}\psi_1(u)v - \tfrac{1}{\mathrm{T}^2}\psi_1(u)v^2 \right] \ln(v + \mathrm{T}) \Bigg\}.
\end{aligned}
\tag{68}
$$

Using Eq. (28), the covariance matrix for the non-stationary part of the flicker noise in phase results in

$$
\begin{aligned}
M^+_{ab} \quad &= h_3 \mathrm{T}^2 \Bigg\{ -\tfrac{1}{2}\delta_{a1}\delta_{b1} - \tfrac{1}{2\mathrm{T}}\delta_{a1}\delta_{b2} - \tfrac{1}{2\mathrm{T}}\delta_{b1}\delta_{a2} - \tfrac{1}{2\mathrm{T}^2}\delta_{a2}\delta_{b2} + \\[2mm]
&\quad \frac{m}{4\pi}\left[\delta_{a1}\sum_{c=1}^{14} L^{-1}_{bc}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left(1 - \tfrac{u^2}{\mathrm{T}^2}\right)\ln(u+\mathrm{T})\,du + \delta_{b1}\sum_{c=1}^{14} L^{-1}_{ac}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left(1 - \tfrac{u^2}{\mathrm{T}^2}\right)\ln(u+\mathrm{T})\,du \right] \\[2mm]
&\quad + \frac{m}{2\pi\mathrm{T}}\left[\delta_{a2}\sum_{c=1}^{14} L^{-1}_{bc}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left(1 + \tfrac{u}{\mathrm{T}}\right)\ln(u+\mathrm{T})\,du + \delta_{b2}\sum_{c=1}^{14} L^{-1}_{ac}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left(1 + \tfrac{u}{\mathrm{T}}\right)\ln(u+\mathrm{T})\,du \right] \Bigg\}.
\end{aligned}
\tag{69}
$$

It is clear from (69) that the only non-zero elements of the matrix M^+_{ab} are $M^+_{1b} = M^+_{b1}$, $M^+_{2b} = M^+_{b2}$, $(b = 1, ..., 14)$. In particular, the elements M^+_{11} and M^+_{22} which contribute to variances first and second parameters are given

$$M^+_{11} = h_3 \mathrm{T}^2 \left[\ln(2\mathrm{T}) - \frac{1991}{1680} \right], \tag{70}$$

$$M^+_{22} = h_3 \left[2\ln(2\mathrm{T}) + \frac{143}{1680} \right]. \tag{71}$$

The stationary part of the covariance matrix M^-_{ab} is given in Tables (11) and (12). Sum of matrices M^+_{ab} and M^-_{ab} gives positive numerical values to variances of all measured parameters.

Flicker noise in frequency, if it dominates in timing residuals, will not improve the accuracy in determination of the initial rotational phase and frequency. Moreover, variance of the initial rotational phase grows as span of observations increases. Information about numerical value of the initial rotational phase will be completely lost when the root square of the variance

when the ratio of red timing noise to the deterministic signal (4) exeeds a certain level. The stage at which this happens may be found from comparision of the mean values of the parameters and their variances.

Table 11. Elements of the covariance matrix M_{ab}^{-} of pulsar's parameters for flicker noise in frequency. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_3 is omitted.

	1	3	5	7	9	11	13
1	$-\frac{79T^2}{3584}$	.	.	.	.	.	.
3	$-\frac{5799}{8960} + \frac{1}{2}\ln(2T)$	$\frac{343}{128T^2}$	.	.	.	.	.
5	$\frac{171}{512T^2}$	$-\frac{441}{256T^4}$	$\frac{735}{512T^6}$	.	.	.	.
7	$\frac{1553Q}{2240}$	$-\frac{189Q}{32T^2}$	$\frac{399Q}{64T^4}$	$\frac{9\pi}{4T}$	.	.	.
9	$-\frac{79Q}{448}$	$\frac{49Q}{32T^2}$	$-\frac{105Q}{64T^4}$	$-\frac{12}{T^2}$	$\frac{15}{4T^2}$	.	.
11	$-\frac{1395Q}{448T^2}$	$\frac{847Q}{32T^4}$	$-\frac{1785Q}{64T^6}$	$-\frac{15\pi}{4T^3}$	$\frac{105}{2T^4}$	$\frac{45\pi}{4T^5}$	.
13	$\frac{79Q}{192T^2}$	$-\frac{343Q}{96T^4}$	$\frac{245Q}{64T^6}$	$\frac{28}{T^4}$	$-\frac{35}{4T^4}$	$-\frac{245}{2T^6}$	$\frac{245}{12T^6}$

Table 12. Elements of the covariance matrix M_{ab}^{-} of pulsar's parameters for flicker noise in frequency. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_3 is omitted.

	2	4	6	8	10	12	14
2	$\frac{27241}{17920} - \ln(2T)$	.	.	.	.	.	.
4	$-\frac{657}{256T^2}$	$\frac{819}{128T^4}$	.	.	.	.	.
6	$\frac{3267}{2560T^4}$	$-\frac{5313}{1280T^6}$	$\frac{7623}{2560T^8}$	.	.	.	.
8	$-\frac{4Q}{35}$	$\frac{39Q}{80T^2}$	$-\frac{33Q}{80T^4}$	$\frac{3}{28}$	.	.	.
10	$\frac{3981Q}{224T^2}$	$-\frac{75Q}{4}$	$\frac{2013Q}{32T^6}$	$-\frac{405}{28T^2}$	$\frac{75\pi}{4T^3}$	.	.
12	$\frac{4Q}{7T^2}$	$-\frac{39Q}{16T^4}$	$\frac{33Q}{16T^6}$	$-\frac{15}{28T^2}$	$\frac{2025}{28T^4}$	$\frac{75}{28T^4}$	.
14	$-\frac{3725Q}{96T^4}$	$\frac{1309Q}{8T^6}$	$-\frac{4389Q}{32T^8}$	$\frac{125}{4T^4}$	$-\frac{105\pi}{4T^5}$	$-\frac{625}{4T^6}$	$\frac{175\pi}{4T^7}$

becomes equal to 2π. Taking into account the non-stationary component (70) of the covariance matrix we find that it happens after amount of orbital revolutions N exeeds $\sim 2/(\nu h_3^{1/2})$. For single pulsars, this corresponds to a time interval of about $\sim 2/(\nu h_3^{1/2})$ years after which the initial rotational phase becomes the non-informative parameter. We also note that variances of the rotational frequency ν and initial orbital phase σ can not be determined better than their values obtained before the flicker noise in frequency has commenced to corrupt observations. Symmetry in variances of corresponding pairs of odd and even orbital parameters is completely destroyed. One interesting consequence of this phenomena is that the measurement precision of orbital parameters in the presence of flicker noise crucially depends on numerical value of the initial orbital phase until it remains to be informative parameter. For example, from equations (44) and (52) one can see that if we arrange observational data to have $\sigma \simeq 0$ then parameter $\dot{x}$ is determined better than the orbital frequency n_b. Inversly, if observational data are prepared to make $\sigma \simeq \pi/2$ then the rotational frequency may be measured better than $\dot{x}$. The situation looks similar to that existing in the quantum mechanical non-demolition experiments (Braginsky & Khalili 1992) in which the quantum

Table 14. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for random walk in frequency. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_4 is omitted.

	2	4	6	8	10	12	14
2	$-\frac{525T}{9152}$	.	.	.	.	.	.
4	$-\frac{1295}{4576T}$	$\frac{1015}{2288T^3}$	.	.	.	.	.
6	$\frac{77}{832T^3}$	$-\frac{105}{416T^5}$	$\frac{693}{4160T^7}$	.	.	.	.
8	$-\frac{15TQ}{2288}$	$\frac{29Q}{1144T}$	$-\frac{21Q}{1040T^3}$	$\frac{3T}{715}$	.	.	.
10	$\frac{105Q}{104T}$	$-\frac{50Q}{13T^3}$	$\frac{315Q}{104T^5}$	$-\frac{15}{26T}$	$\frac{5325}{52T^3}$	.	.
12	$\frac{75Q}{2288T}$	$-\frac{145Q}{1144T^3}$	$\frac{21Q}{208T^5}$	$-\frac{3}{143T}$	$\frac{75}{26T^3}$	$\frac{15}{143T^3}$	.
14	$-\frac{315Q}{143T^3}$	$\frac{4795Q}{572T^5}$	$-\frac{343Q}{52T^7}$	$\frac{357}{286T^3}$	$-\frac{10815}{52T^5}$	$-\frac{1785}{286T^5}$	$\frac{250915}{572T^7}$

Random walk in frequency also converts the initial orbital phase or semi-major axis of the binary pulsar from the informative to the non-informative parameters. Indeed, equations (43), (47) and variances of parameters β_7, β_8 taken from Tables 13, 14 yield

$$< (\delta\sigma)^2 > \quad = \quad \frac{h_4}{x^2} \left(\frac{2493}{572T} \cos^2 \sigma + \frac{3T}{715} \sin^2 \sigma \right) , \tag{75}$$

$$< (\delta x)^2 > \quad = \quad h_4 \left(\frac{3T}{715} \cos^2 \sigma + \frac{2493}{572T} \sin^2 \sigma \right) . \tag{76}$$

We can see that if σ has been determined prior to the moment when the random walk in frequency begins to dominate, and its numerical value was not close to $\pi/2$ or $3\pi/2$, then the information about this value is lost after $(2860\pi x^2)/(3h_4) \simeq 2995x^2/h_4$ orbital revolutions. In the case when σ is close to $\pi/2$ or $3\pi/2$ the information will be lost about parameter x. Arranging observations in the interval before random walk begins a dominant source of the noise we can preserve information either about numerical value of σ or x parameters. If the initial orbital phase becomes the non-informative parameter one has to average equations (43)-(50) with respect to σ under assumption that distribution of probability for this parameter is uniform and concentrated in the interval from 0 to 2π (Bard 1974).

7.7   Flicker Noise in Frequency Derivative ($1/f^5$ noise)

The timing residuals are proportional to the forth power of the number of orbital revolution

$$< r^2 > = \frac{1}{8400} h_5 T^4 . \tag{77}$$

The non-stationary part of the autocovariance function for flicker noise in frequency derivative can be expressed through the basic functions as (Kopeikin 1997b)

$$R^+(t_i, t_j) \quad = \quad \frac{1}{16} h_5 T^4 \left\{ -5\psi_1(u)\psi_1(v) - \frac{10}{T} [\psi_2(u)\psi_1(v) + \psi_1(u)\psi_2(v)] - \right. \tag{78}$$

$$\frac{17}{3\mathrm{T}^2}\left[\psi_3(u)\psi_1(v) - \frac{56}{17}\psi_2(u)\psi_2(v) + \psi_1(u)\psi_3(v)\right] \tag{79}$$

$$-\frac{2}{3\mathrm{T}^3}\left[\psi_4(u)\psi_1(v) + 14\psi_3(u)\psi_2(v) + 14\psi_2(u)\psi_3(v) + \psi_1(u)\psi_4(v)\right] \tag{80}$$

$$-\frac{2}{3\mathrm{T}^4}\left[\psi_4(u)\psi_2(v) + \frac{11}{2}\psi_3(u)\psi_3(v) + \psi_2(u)\psi_4(v)\right]\Bigg\} \tag{81}$$

$$+\frac{1}{16}h_5\mathrm{T}^4\left\{2\psi_1(v) + \left[\frac{8}{3\mathrm{T}}u\psi_1(v) + 2\psi_2(v)\right] + \frac{4}{\mathrm{T}^2}\left[2u\psi_2(v) + \psi_3(v)\right] + \frac{8}{\mathrm{T}^3}u\psi_3(v)\right. \tag{82}$$

$$+\frac{2}{3\mathrm{T}^4}\left[u^4\psi_1(v) - 4u^3\psi_2(v) + 6u^2\psi_3(v)\right]\Bigg\}\ln(u+\mathrm{T}) \tag{83}$$

$$+\frac{1}{16}h_5\mathrm{T}^4\left\{2\psi_1(u) + \left[\frac{8}{3\mathrm{T}}v\psi_1(u) + 2\psi_2(u)\right] + \frac{4}{\mathrm{T}^2}\left[2v\psi_2(u) + \psi_3(u)\right] + \frac{8}{\mathrm{T}^3}v\psi_3(u)\right. \tag{84}$$

$$+\frac{2}{3\mathrm{T}^4}\left[v^4\psi_1(u) - 4v^3\psi_2(u) + 6v^2\psi_3(u)\right]\Bigg\}\ln(v+\mathrm{T}) \;. \tag{85}$$

Using (28) the covariance matrix for the non-stationary part of the flicker noise in frequency derivative is reduced to the form

$$M_{ab}^+ \;=\; \frac{1}{16}h_5\mathrm{T}^4\left[-5\delta_{a1}\delta_{b1} - \frac{10}{\mathrm{T}}\delta_{a1}\delta_{b2} - \frac{10}{\mathrm{T}}\delta_{b1}\delta_{a2} - \frac{17}{3\mathrm{T}^2}\delta_{a1}\delta_{b3} - \frac{17}{3\mathrm{T}^2}\delta_{b1}\delta_{a3} + \frac{56}{3\mathrm{T}^2}\delta_{a2}\delta_{b2}\right. \tag{86}$$

$$-\frac{2}{3\mathrm{T}^3}\delta_{a1}\delta_{b4} - \frac{2}{3\mathrm{T}^3}\delta_{b1}\delta_{a4} - \frac{28}{3\mathrm{T}^3}\delta_{a2}\delta_{b3} - \frac{28}{3\mathrm{T}^3}\delta_{b2}\delta_{a3} - \frac{2}{3\mathrm{T}^4}\delta_{a2}\delta_{b4} - \frac{2}{3\mathrm{T}^4}\delta_{b2}\delta_{a4} - \frac{11}{3\mathrm{T}^4}\delta_{a3}\delta_{b3}\right] + \tag{87}$$

$$\frac{m}{2\pi}\left\{\delta_{a1}\sum_{c=1}^{14}L_{bc}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[2\mathrm{T}^4 + \frac{8}{3}\mathrm{T}^3 u + \frac{1}{3}u^4\right]\ln(u+\mathrm{T})du \;+ \tag{88}$$

$$\delta_{b1}\sum_{c=1}^{14}L_{ac}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[2\mathrm{T}^4 + \frac{8}{3}\mathrm{T}^3 u + \frac{1}{3}u^4\right]\ln(u+\mathrm{T})du\right\} \tag{89}$$

$$+\frac{m}{2\pi}\left\{\delta_{a2}\sum_{c=1}^{14}L_{bc}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[\frac{16}{3}\mathrm{T}^3 + 8\mathrm{T}^2 u - \frac{8}{3}u^3\right]\ln(u+\mathrm{T})du \;+ \tag{90}$$

$$\delta_{b2}\sum_{c=1}^{14}L_{ac}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[\frac{16}{3}\mathrm{T}^3 + 8\mathrm{T}^2 u - \frac{8}{3}u^3\right]\ln(u+\mathrm{T})du\right\} \tag{91}$$

$$+\frac{m}{2\pi}\left\{\delta_{a3}\sum_{c=1}^{14}L_{bc}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[4\mathrm{T}^4 + 8\mathrm{T}u + u^2\right]\ln(u+\mathrm{T})du \;+ \tag{92}$$

$$\delta_{b3}\sum_{c=1}^{14}L_{ac}^{-1}\int_{-\mathrm{T}}^{\mathrm{T}}\psi_c(u)\left[4\mathrm{T}^4 + 8\mathrm{T}u + 4u^2\right]\ln(u+\mathrm{T})du\right\} . \tag{93}$$

Table 15. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for flicker noise in frequency derivative. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_5 is omitted.

	1	3	5	7	9	11	13
1	$-\frac{13T^4}{16128}$	.	.	.	.	.	.
3	$\frac{127T^2}{8064}$	$\frac{1039}{6720} - \frac{1}{4}\ln(2T)$	.	.	.	.	.
5	$\frac{3589}{241920} - \frac{1}{24}\ln(2T)$	$-\frac{19}{128T^2}$	$\frac{21}{256T^4}$	.	.	.	.
7	$\frac{1033T^2Q}{20160}$	$-\frac{307Q}{1120}$	$\frac{15Q}{64T^2}$	$\frac{303}{280}$	.	.	.
9	$-\frac{13T^2Q}{1008}$	$\frac{Q}{14}$	$-\frac{Q}{16T^2}$	$-\frac{17}{56}$	$\frac{5}{56}$	.	.
11	$-\frac{929Q}{4032}$	$\frac{275Q}{224T^2}$	$-\frac{67Q}{64T^4}$	$-\frac{269}{56T^2}$	$\frac{75}{56T^2}$	$\frac{1195}{56T^4}$	.
13	$\frac{13Q}{432}$	$-\frac{Q}{6T^2}$	$\frac{7Q}{48T^4}$	$\frac{17}{24T^2}$	$-\frac{5}{24T^2}$	$-\frac{25}{8T^4}$	$\frac{35}{72T^4}$

Table 16. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for flicker noise in frequency derivative. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_5 is omitted.

	2	4	6	8	10	12	14
2	$-\frac{163T^2}{9216}$	.	.	.	.	.	.
4	$-\frac{65657}{483840} + \frac{1}{6}\ln(2T)$	$\frac{93}{256T^2}$	.	.	.	.	.
6	$\frac{1243}{15360T^2}$	$-\frac{429}{2560T^4}$	$\frac{2541}{25600T^6}$	.	.	.	.
8	$-\frac{163T^2Q}{40320}$	$\frac{31Q}{2240}$	$-\frac{33Q}{3200T^2}$	$\frac{T^2}{560}$	.	.	.
10	$\frac{2521Q}{4032}$	$-\frac{471Q}{224T^2}$	$\frac{99Q}{64T^4}$	$-\frac{1}{4}$	$\frac{1005}{28T^2}$	.	.
12	$\frac{163Q}{8064}$	$-\frac{31Q}{448T^2}$	$\frac{33Q}{640T^4}$	$-\frac{1}{112}$	$\frac{5}{4T^2}$	$\frac{5}{112T^2}$	.
14	$-\frac{131Q}{96T^2}$	$\frac{55Q}{12T^4}$	$-\frac{539Q}{160T^6}$	$\frac{13}{24T^2}$	$-\frac{935}{12T^4}$	$-\frac{65}{24T^4}$	$\frac{1015}{6T^6}$

It is clear from (62) that the only elements of the matrix M_{ab}^+ being not equal to zero are $M_{1b}^+ = M_{b1}^+$, $M_{2b}^+ = M_{b2}^+$, and $M_{3b}^+ = M_{b3}^+$, where the index $b = 1, ..., 14$. In particular, the elements M_{11}^+, M_{22}^+, and M_{33}^+ are given

$$M_{11}^+ = \frac{1}{4}h_5 T^4 \left[\ln(2T) - \frac{613}{315}\right] , \tag{94}$$

$$M_{22}^+ = h_5 T^2 \left[\ln(2T) - \frac{2999}{2520}\right] , \tag{95}$$

$$M_{33}^+ = \frac{1}{2}h_5 \left[\ln(2T) + \frac{307}{840}\right] . \tag{96}$$

The matrix M_{ab}^- is given in Tables (15)-(16). From these tables, equations (43)-(50), and discussions of previous sections we conclude that just the flicker noise in frequency becomes a dominant source of noise in timing residuals - the initial rotational

phase $\mathcal{N}_0$, rotational frequency ν, the initial orbital phase σ, the semi-major axis x - are getting the non-informative parameters after a certain period of time which can be calculated in the same way as it has been done in previous sections. Determination of numerical values of parameters - $\dot{\nu}$, $\dot{x}$, and the orbital frequency n_b can not be improved comparatively to those values which was obtained at preceding epoch when timing noise was not so red. This has a consequence for stability of the, so called, binary pulsar time (BPT) scale (Ilyasov, Kopeikin & Rodin 1998) which is discussed in section 8. Flicker noise in frequency may be produced by the stochastic gravitational wave background. Detailed discussion of the analysis of timing data from binary pulsars in the presence of this noise is given by Kopeikin (1997a).

7.8 Random Walk in Frequency Derivative ($1/f^6$ noise)

The timing residuals are proportional to the fifth power of the number of orbital revolutions

$$< r^2 >= \frac{4}{765765} h_6 \mathrm{T}^5, \tag{97}$$

where h_6 is a constant quantity characterizing intensity of the noise. The non-stationary part of the autocovariance function for PN random walk is represented as (Kopeikin 1997b)

$$R^+(t_i, t_j) = \frac{1}{20} h_6 \mathrm{T}^5 \left\{ \psi_1(u)\psi_1(v) + \frac{5}{2\mathrm{T}} \left[\psi_1(u)\psi_2(v) + \psi_2(u)\psi_1(v) \right] + \right. \tag{98}$$

$$+ \frac{5}{3\mathrm{T}^2} \left[\psi_1(u)\psi_3(v) + 4\psi_2(u)\psi_2(v) + \psi_3(u)\psi_1(v) \right] \tag{99}$$

$$+ \frac{5}{\mathrm{T}^3} \left[\psi_3(u)\psi_2(v) + \psi_2(u)\psi_3(v) \right] + \frac{5}{\mathrm{T}^4} \psi_3(u)\psi_3(v) \tag{100}$$

$$+ \frac{1}{12\mathrm{T}^5} \left[\psi_1(u)\psi_6(v) - 5\psi_2(u)\psi_5(v) + 10\psi_3(u)\psi_4(v) \right. \tag{101}$$

$$\left. + 10\psi_4(u)\psi_3(v) - 5\psi_5(u)\psi_2(v) + \psi_6(u)\psi_1(v) \right] \right\} . \tag{102}$$

Using formula (27) the covariance matrix for non-stationary part of PN is reduced to the form

$$M_{ab}^+ = \frac{1}{20} h_6 \mathrm{T}^5 \left[\delta_{a1}\delta_{b1} + \frac{5}{2\mathrm{T}} \delta_{a1}\delta_{b2} + \frac{5}{2\mathrm{T}} \delta_{b1}\delta_{a2} + \frac{20}{3\mathrm{T}^2} \delta_{a2}\delta_{b2} \right. \tag{103}$$

$$+ \frac{5}{3\mathrm{T}^2} \delta_{a1}\delta_{b3} + \frac{5}{3\mathrm{T}^2} \delta_{b1}\delta_{a3} + \frac{5}{\mathrm{T}^3} \delta_{a2}\delta_{b3} + \frac{5}{\mathrm{T}^3} \delta_{b2}\delta_{a3} + \frac{5}{\mathrm{T}^4} \delta_{a3}\delta_{b3} \tag{104}$$

$$+ \frac{1}{12\mathrm{T}^5} \left(\delta_{a1}\delta_{b6} + \delta_{b1}\delta_{a6} - 5\delta_{a2}\delta_{b5} - 5\delta_{b2}\delta_{a5} + 10\delta_{a3}\delta_{b4} + 10\delta_{b3}\delta_{a4} \right) \right]. \tag{105}$$

The stationary part of the covariance matrix M_{ab}^- is given in Tables (17)-(18). Random walk in frequency is so strong at low frequencies that time derivatives of rotational phase also becomes the non-informative parameter after a certain period of time. Neither the orbital phase, nor the semi-major axis can be measured precisely. Variances of all parameters which depend

Table 17. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for random walk in frequency derivative. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_6 is omitted.

	1	3	5	7	9	11	13
1	$-\frac{43T^5}{480480}$	.	.	.	.	.	.
3	$\frac{53T^3}{41184}$	$-\frac{35T}{1716}$	.	.	.	.	.
5	$-\frac{3T}{572}$	$-\frac{105}{4576T}$	$\frac{35}{4576T^3}$	.	.	.	.
7	$\frac{25T^3Q}{3432}$	$-\frac{Q}{44}$	$\frac{19Q}{1144T}$	$\frac{9T}{143}$	.	.	.
9	$-\frac{13T^2Q}{1008}$	$\frac{Q}{14}$	$-\frac{Q}{16T^2}$	$-\frac{17}{56}$	$\frac{5}{56}$	.	.
11	$-\frac{263TQ}{8008}$	$\frac{175Q}{1716T}$	$-\frac{85Q}{1144T^3}$	$-\frac{40}{143T}$	$\frac{155}{2002T}$	$\frac{1245}{1001T^3}$	.
13	$\frac{43TQ}{10296}$	$-\frac{35Q}{2574T}$	$\frac{35Q}{3432T^3}$	$\frac{35}{858T}$	$-\frac{5}{429T}$	$-\frac{155}{858T^3}$	$\frac{35}{1287T^3}$

Table 18. Elements of the covariance matrix M_{ab}^- of pulsar's parameters for random walk in frequency derivative. Quantities $T = \pi N$ and $Q = \cos T$. The magnitude of noise h_6 is omitted.

	2	4	6	8	10	12	14
2	$-\frac{581T^3}{700128}$	.	.	.	.	.	.
4	$\frac{861T}{77792}$	$\frac{105}{2431T}$	.	.	.	.	.
6	$\frac{189}{17680T}$	$-\frac{7}{544T^3}$	$\frac{231}{35360T^5}$	.	.	.	.
8	$-\frac{83T^3Q}{291720}$	$\frac{2TQ}{2431}$	$-\frac{Q}{1768T}$	$\frac{T^3}{12155}$	.	.	.
10	$\frac{865TQ}{19448}$	$-\frac{1225Q}{9724T}$	$\frac{151Q}{1768T^3}$	$-\frac{57T}{4862}$	$\frac{375}{221T}$	.	.
12	$\frac{83TQ}{58344}$	$-\frac{10Q}{2431T}$	$\frac{5Q}{1768T^3}$	$-\frac{T}{2431}$	$\frac{285}{4862T}$	$\frac{5}{2431T}$	.
14	$-\frac{17003Q}{175032T}$	$\frac{8015Q}{29172T^3}$	$-\frac{329Q}{1768T^5}$	$\frac{371}{14586T}$	$-\frac{8960}{2431T^3}$	$-\frac{1855}{14586T^3}$	$\frac{1225}{153T^5}$

on the numerical value of the initial orbital phase σ should be averaged with respect to σ assuming the uniform distribution of probability density function in the interval from 0 to 2π.

8 KINEMATIC AND DYNAMIC PULSAR TIME SCALES

In this section we discuss stability of time scales based on kinematic rotation of pulsars (PT scale) and orbital motion of pulsar in a binary system (BPT scale). This comprehensive analysis can be done due to using results of calculation of timing residuals and covairance matrices given in the preceding sections.

The methodology of applying pulsar timing data to fundamental metrology and time keeping service was explicitly formulated by Russian astronomers in 1979 (Shabanova *et al.* 1979, see also Ilyasov *et al.* 1989). Subsequent timing observations definitely proved consistency of this approach and made it clear that PT scale has a stability comparable with that of atomic clocks being placed in metrological centres (Ilyasov *et al.* 1989, Ginot and Petit 1991, Kaspi *et al.* 1994). However, it does not

mean that one should stop looking for other natural periodic physical phenomena which might be used for generating time scales being even more stable than PT. A new possibility to explore the problem is creation and maintenance of modified ephemeris time (ET) of classical astronomy based not on the orbital motion of planets in the Solar system (Guinot 1989) but on the orbital motion of pulsar in a close binary system (Petit and Tavella 1996, Ilyasov *et al.* 1998). It has been theoretically justified (Damour 1987, Kopeikin 1985, Schäfer 1985) and observationally confirmed with accuracy 0.4% (Damour and Taylor 1991, 1992) that orbital motion of pulsar in a binary system is governed by laws of General Relativity. For this reason one can predict the orbital motion of pulsar with extremely high precision on very long time intervals larger than 10 years generating in this way the new time scale BPT. However, the principal question arises about the limit on the stability of such time scale in the presence of red stochastic noise process in pulsar timing residuals. We answer this question in the present section.

8.1   Binary pulsars as time keeping standards

Standards of time and frequency have three important characteristics: (1) frequency stability, (2) reproduction of the time scale unit, and (3) the span of life time. Regarding these characteristics we note that the life time of binary pulsars reaches 10^6 years and more which makes them very long-lived time standards. Orbital motion of pulsar in a close binary system is practically not subjected to external gravitational perturbations. For this reason the orbital motion has a very high intrinsic stability allowing reproduction of duration of one orbital revolution to very high precision. Binary pulsar time scale - BPT (Petit & Tavella 1996, Ilyasov *et al.* 1998) can be constructed according to the formula

$$\mathcal{T} \quad = \quad \mathcal{T}_0 + P_b N + \frac{1}{2}\dot{P}_b N^2, \tag{106}$$

where $\mathcal{T}_0$ is an initial epoch for orbital phase, N is the number of orbital revolutions, P_b and $\dot{P}_b$ are the orbital period and its time derivative reffered to the initial epoch $\mathcal{T}_0$. This equation does not contain terms depending on high-order time derivatives of the orbital period as they equal identically to zero in the barycentric reference frame of the binary system (Kopeikin 1985, Kopeikin and Potapov 1994). However, radial acceleration of the binary system with respect to observer and its proper motion in the sky bring about appearance of the second and next high-order time derivatives both of P_b (Damour & Taylor 1992, Bell & Bailes 1996) and of the projected semi-major axis of the pulsar's orbit (Kopeikin 1996, 1997a)(see equation (4) for more detail). These effects change the intrinsic values of P_b and $\dot{P}_b$ and make more difficult their determination from observations.

   An additional serious problem arises with the possible presence in TOA the low-frequency noise of astrophysical origin. Indeed, intrinsic rotational motion of pulsar may be perturbed by the random walk in phase and/or its time derivatives (Cordes & Greenstein 1981), interstellar medium and relic stochastic gravitational waves change randomly propagation of electromagnetic pulses from the pulsar to observer being the reason for contamination of timing residuals by flicker noise (Mashhoon & Grishchuk 1980, Bertotti *et al.* 1983, Blandford *et al.* 1984, Backer & Foster 1990, Kopeikin 1997a). As a result, timing residuals represent a pure deterministic process upon which the stohastic additive noise is imposed. This noise consists of mixture of white noise of errors of observations and the low-frequency red noise of astrophysical origin. The spectrum $S(f)$ of the red noise is described in Table 2. It contains both stationary and non-stationary components (Kopeikin 1997b). The

presence of the red noise makes specific restrictions on the stability of both PT and BPT time scales which we briefly discuss in what follows. More detail on the subject may be found in the paper by Ilyasov, Kopeikin & Rodin (1998).

8.2 BPT scale and its stability

A complete treatment of the problem under consideration requires rather combersome calculations. In order to avoid mathematical difficulties we restrict ourselves to a simplified model of a binary system with the pulsar on the circular orbit so that all results of previous sections can be applied without any restrictions. Our goal is to get the optimal estimate of measured stability of BPT on the background of additive, low-frequency red noise in TOA. This problem had been partially considered (see, for instance, Van Trees 1968, Tikhonov 1983). Unfortunately, the methods developed in most previous publications can not be applied directly in our study so that we need to rely upon our own approach. In the problem under consideration the signal represents a linear combination of rotational phase of the pulsar, which is given by a polynomial of time, and quasi-periodic sinusoidal function with the argument being the orbital phase which is also a polynomial of time. As we have assumed the noise is additive to the signal, we can use the functional expression for the analyzed signal in the form:

$$\xi(t) \quad = \quad \mathcal{N}(t, \beta_a) + \nu\, \epsilon(t), \quad 0 \le t \le \tau\,, \tag{107}$$

where $\mathcal{N}(t, \beta_a)$ is function of time describing deterministic part of the signal, depending on measured parameters β_a, and given by equation (4); ν is the pulsar rotational frequency; $\epsilon(t)$ is a random process defined by equation (5) which differentiate between noises which are true instabilities of the pulsar clock, and apparent instabilities that arise from random galactic accelerations, gravitational waves, etc. The autocorrelation function of the noise process $\epsilon(t)$ is given in Table 2, τ is the total span of observational time. We also assume that observations are evenly spaced along the orbit with frequency m. It has allowed us to replace in all sums by integrals and to treat the stochastic process $\epsilon(t)$ as continuous.

For an adequate treatment of stability of time scales PT and BPT it is convenient to introduce two new parameters $y = \delta\nu/\nu$ and $v = \delta n_b/n_b$, where $\delta\nu$ and δn_b represent differences between true physical value of the parameter in question and its estimate obtained by the least squares method when fitting TOA of the pulsar. Variances of estimates of rotational and orbital frequencies (σ_y^2, and, σ_v^2, respectively) are convenient for comparing stability of two time scales - dynamic BPT and kinematic PT. Mathematical expressions for these variances are called true variances (Rutman 1978) and represent the close analogue of the Allan variance being used in fundamental metrology for analysing stability of time-frequency standards. Expressions for σ_v and σ_y crucially depend on the type of noise present and can be evaluated from the variance of β_2 and from equation (44) and variances β_7, β_8. Their explicit dependence on the amplitude of noise h_s, $s = 0,\ 1,\ 2,\dots,\ 6$ and duration of the total span of observations τ are given in Table 19 and are plotted schematically in Fig. 2 where the horizontal axis shows the total span of observations τ and vertical axis numerates values of the parameters σ_y and σ_v. It is worth emphasizing that σ_y can not depend on orbital parameters of the pulsar because it characterizes instability of the intrinsic rotational frequency of the pulsar and, for this reason, can not be related with its orbital motion. On the other hand, the quantity σ_v naturally depends on the orbital period of the binary $P_b = 2\pi/n_b$ and its projected semi-major axis x since these parameters characterize frequency and amplitude of the orbital sine wave in timing residuals. Additional functional dependence of σ_v
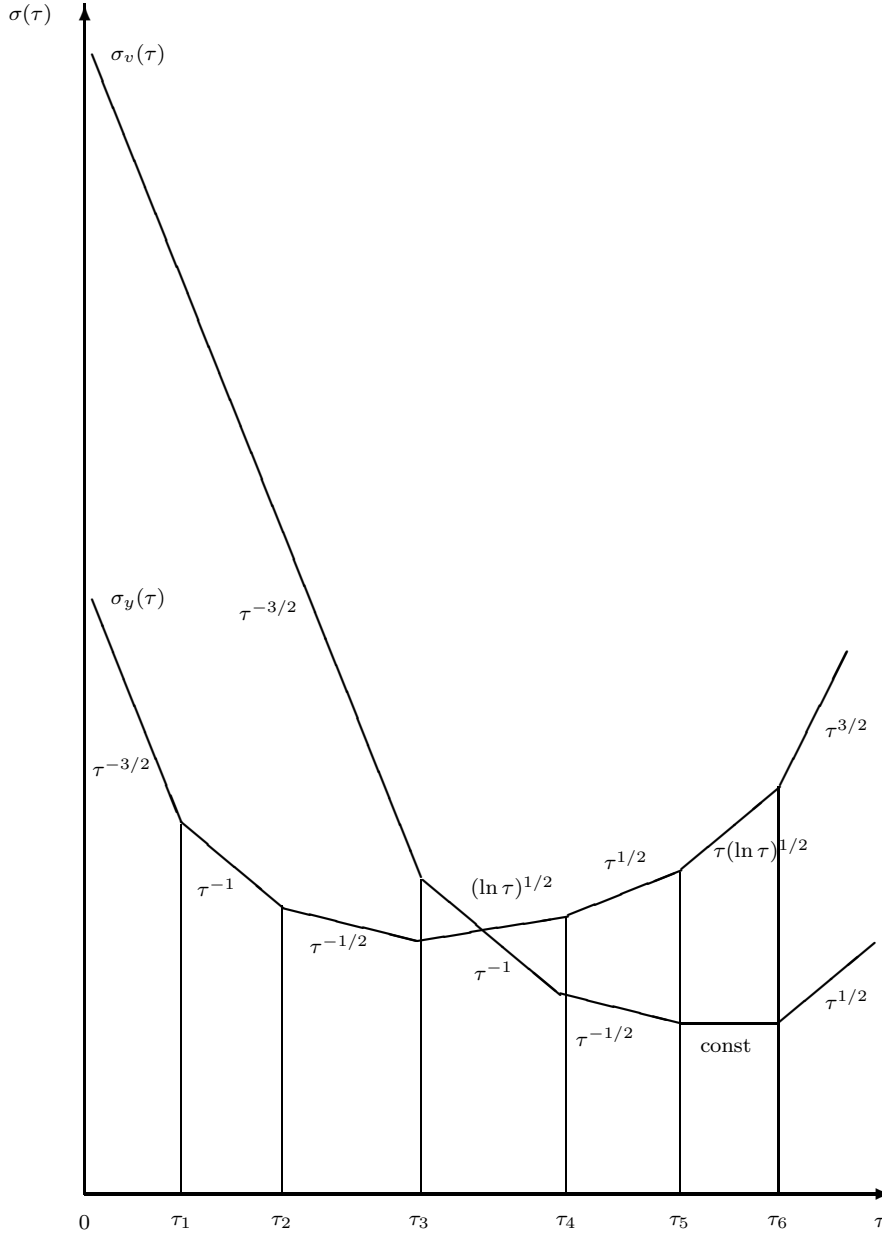
Figure 2. Characteristic behavior of variances characterizing stability of intrinsic rotational frequency of pulsar σ_y and that of the orbital frequency of the binary system σ_v. In the case of evenly spaced observations theoretically predicted ratio σ_y/σ_v is equal to $1.75\pi\, x/P_b \simeq 10^{-3} \div 10^{-4}$ until white noise dominates in timing residuals. Displacement of two curves in subsequent moments of time depends on the relative magnitude of white and red noise components.

on the initial orbital phase σ in the case of random walk in phase is explained by the fact that the orbital frequency of the binary is one of the quadrature component of the sine function (the second quadrature component is the first time derivative of the projected semi-major axis $\dot{x}$ as follows from the definition of parameters β_9 and β_{10} given in Table 1). Low-frequency red noise with spectral index $s = 2$ distroys an equilibrium in the accuracy of simultaneous determination of the quadrature components and leads to the dependence of σ_v on the parameter σ. In the case of red noise with index $s \geq 3$ the initial orbital phase is the non-informative parameter. Thus, we have averaged the variances with respect to σ which, for this reason, does not appear in the corresponding expressions for σ_v.

In constructing Fig. 2 we have assumed that during the time interval $[\tau_0, \tau_1]$ white phase noise dominates. This noise is caused by the presence of errors of measurements in TOA and, possibly, by the uncorrelated noise in the intrinsic rotation of the pulsar. Various components of the low-frequency red noise commence to appear on time intervals $\tau > \tau_1$. Their amplitudes h_s, as the experience of working with atomic clocks shows, gradually decrease as the spectral index s of the noise increases. For instance, in the model being considered in Fig. 2 the red flicker noise with spectral density $S(f) \simeq h_1/f$ dominates in timing residuals from the moment τ_1 till τ_2. Then, starting from the moment τ_2 up to τ_3 the random walk in phase with the spectral density $S(f) \simeq h_2/f^2$ dominates, and so on. In general, we assume that the longer time interval of observations the more significant the contribution of the red noise with higher numerical value of the spectral index s. Such behavior relates to the fact that usually the red noise with higher spectral index has lower intensity (that is, smaller amplitude h_s) and can give rise to a noticable contribution only on longer time interval τ from the beginning of observation.

It is interesting to note that behavior of $\sigma_y(\tau)$ is reminiscent of the, so-called, narrow-band frequency dispersion used in the fundamental metrology for characterizing stability of time-frequency standards when low-frequency noise is dominating (Rutman, 1978). However, in the timing model under consideration, $\sigma_y(\tau)$, turns out to be significantly dependent on the non-stationary part of red noise with the spectral indices $s = 3$ or high (see Table 19). As there is no satisfactory theoretical model of the non-stationary component such dependence of $\sigma_y(\tau)$ is an indication of necessity of working out the other, more practical approach for estimation of the variance characterizing stability of intrinsic rotational motion of pulsar and PT scale. It seems that the modified measure of the variance, which is called σ_z instead of σ_y, introduced by Taylor (1992) and based on the using of mathematical technique of transfer filter functions in spectral frequency domain is rather constructive and fruitful step ahead toward this direction. In particular, Matsakis *et al.* (1997) have recently derived the explicit expression for $\sigma_z(\tau)$ which is proportional to the second time derivative of pulsar's rotational frequency $\dot{\nu}$. We have used our analytic estimation of pulsar parameters in timing model (4) in order to calculate σ_z. Results of the calculations is shown in Table 19 along with σ_y and σ_v. We confirm the statement of Matsakis *et al.* (1997) that function σ_z is independent from the non-stationary component of red noise. Moreover, it does not contain slow logarithmic trends explicitly appearing in σ_y for flicker noises with indices $s = 3$ and $s = 5$.

Comparing the σ_z statistic with σ_v we find that σ_v is not sensitive to the non-stationary component of red noise as well as σ_z, and does not contain the logarifmic trends. However, σ_v may depend on the initial orbital phase for red noises with indices $s \geq 2$ if span of observational time is not long enough to make that parameter to be non-informative. Our calculations also clearly show that measured value of the variance $\sigma_v(\tau)$ is not sensitive to red noise with index $s \leq 2$ and, for this reason, does not allow one to distingiush such red noise from the white one. Nevertheless, this specific behavior of $\sigma_v(\tau)$ opens a possibility of using orbital motion of the pulsar as a new time reference being more stable than PT scale on longer intervals. Indeed, as one can see in Fig. 2, function $\sigma_y(\tau)^{\star\star}$ describing stability of PT scale begins to grow from the moment τ_3 but $\sigma_v(\tau)$ characterizing stability of BPT scale continues to decrease until the red noise with spectral index $s \geq 5$ begins to dominate.

** Behavior of σ_z is more representative for characterizing stability of PT scale. However, graphic behavior of σ_y looks approximately the same as that of σ_z.

Table 19. Dependence of variances $\sigma_y(\tau)$, $\sigma_z(\tau)$, and $\sigma_v(\tau)$ on the total span of observations τ for red noise with different spectra. Herein, h_s, $(s = 1, 2 \ldots, 6)$ is the amplitude of the corresponding noise component; P_b is the orbital period and x is the projected semi-major axis of the binary orbit, σ is the initial orbital phase, Δt is the interval of time between two successive observations, $C_3, ..., C_6$ are positive constants of order 0.05 - 0.5 reflecting dependence of the variances on the non-stationary component of the noise model.

$S(f)$	$\sigma_y^2(\tau)$	$\sigma_z^2(\tau)$	$\sigma_v^2(\tau)$
h_0	$\frac{3675}{16}\Delta t\, h_0\, \tau^{-3}$	$\frac{2835}{16}\Delta t\, h_0\, \tau^{-3}$	$\frac{75}{2\pi^2}\frac{P_b^2}{x^2}\Delta t\, h_0\, \tau^{-3}$
h_1/f	$\frac{4851}{64}h_1\tau^{-2}$	$\frac{2499}{64}h_1\tau^{-2}$	$\frac{75}{4\pi^2}\frac{P_b^3}{x^2}h_1\tau^{-3}$
h_2/f^2	$\frac{1575}{416}h_2\tau^{-1}$	$\frac{441}{416}h_2\tau^{-1}$	$\frac{1275}{88\pi^4}\frac{P_b^4}{x^2}(\sin^2\sigma + \frac{11}{17}\cos^2\sigma)h_2\tau^{-3}$
h_3/f^3	$(C_3 + \ln\tau)h_3$	$\frac{819}{2560}h_3$	$\frac{15}{32\pi^4}\frac{P_b^4}{x^2}h_3\tau^{-2}$
h_4/f^4	$(C_4 - \frac{525}{18304})h_4\tau$	$\frac{203}{18304}h_4\tau$	$\frac{25}{2288\pi^4}\frac{P_b^4}{x^2}h_4\tau^{-1}$
h_5/f^5	$\frac{1}{4}(C_5 + \ln\tau)h_5\tau^2$	$\frac{93}{20480}h_5\tau^2$	$\frac{5}{1792\pi^4}\frac{P_b^4}{x^2}h_5$
h_6/f^6	$(C_6 - \frac{581}{5601024})h_6\tau^3$	$\frac{21}{77792}h_6\tau^3$	$\frac{5}{64064\pi^4}\frac{P_b^4}{x^2}h_6\tau$

This relative behavior of two variances is quite general and does not depend on the specific numerical value of intensities of red noise. Theoretical analysis reveals, that the minimum of $\sigma_v(\tau)$ is reached significantly later than that of $\sigma_y(\tau)$ (and $\sigma_z(\tau)$ respectively). Depth of the minimum for $\sigma_y(\tau)$ (and $\sigma_z(\tau)$ depends on the noise with spectral index $s = 3$ which is produced by the large-scale inhomogeneities of the interstellar medium (Blandford *et al.* 1984). Depth of the minimum for $\sigma_v(\tau)$ depends on the amplitude of noises with spectral indecies $s = 5$ which exist due to the stohastic background of relic gravitational radiation produced by physical processes in early universe (Mashhoon & Grishchuk 1980, Bertotti *et al.* 1983, Kopeikin 1997a). In principle, noise with spectral index $s = 5$ may arise also as a result of random fluctuations of the first time derivative of intrinsic rotational frequency of the pulsar (Cordes & Greenstein 1981). The existence of such fluctuations is extremely unlikely and, for this reason, they are not discussed in the following discussion. As an example we specify the numerical value of the minimum of $\sigma_z(\tau)$ for a single millisecond pulsar PSR B1937+21 which reach 10^{-14} on time interval 2-3 years. Existence of the minimum with subsequent turning-up of the curve $\sigma_z(\tau)$ for this pulsar is explained by the dominance of random instabilities in the rotational phase of the pulsar (Kaspi *et al.* 1994).

It is remarkable from principal and practical points of view that depth of the absolute minimum for the function, $\sigma_v(\tau)$, can be predicted with a high degree of certainty. It is defined by the amplitude of the relic stochastic gravitational wave background radiation with spectral index $s = 5$. More specifically, for binary pulsars with circular orbits the depth of minimum of $\sigma_v(\tau)$ is defined by the expression:

$$\sigma_v \simeq 1.2 \cdot 10^{-20}\sqrt{\Omega_g}P_b^2 x^{-1}\text{h}, \tag{108}$$

where Ω_g is energy density of stochastic gravitational wave background radiation per logarithmic unit interval of frequency,

P_b is the orbital period of the binary measured in seconds, x is the projected semi-major axis of the pulsar's orbit measured in seconds, h is the Hubble constant in units 100 km $\mathrm{c}^{-1}\mathrm{Mpc}^{-1}$. It is worthwhile to point out (as the formula (108) shows) that numerical value of the minimum for $\sigma_v(\tau)$ depends not only on the magnitude of fundamental cosmological parameters Ω_g and h but on the numerical values of orbital parameters of the binary system as well. Of course, for binary pulsars with elliptical orbits formula (5) will also include dependence on the eccentricity of the orbit. We shall consider this more complicated case elsewhere.

Formula (108) yields the absolute (fundamental) value of the minimum for $\sigma_v(\tau)$. However, the real minimum of $\sigma_v(\tau)$ depends significantly on the amplitude of noise with spectral index $s = 6$. If the amplitude of this noise is less than that of the stochastic noise of gravitational wave backround radiation, then the real minimum of $\sigma_v(\tau)$ coincides with absolute minimum given by formula (108). In the opposite case the minimum of $\sigma_v(\tau)$ will be located above the absolute minimum (108). In any case, the observed numerical value of minimum for $\sigma_v(\tau)$ may, in principle, reach the level $10^{-15} \div 10^{-16}$ under assumption that $\Omega_g h^2 \leq 10^{-8}$ (Starobinsky 1979, Vilenkin 1981, Rubakov *et al.* 1982). The result obtained can be used in two ways. First, existence of fundamental minimum for $\sigma_v(\tau)$ allows to search for (or to establish the upper limit on) the stochastic gravitational wave radiation by means of using binary pulsars with long orbital periods and large ratio of P_b/x which is necessary for reducing the absolute value of minimum of $\sigma_v(\tau)$ as much as possible. That will reduce the interval of time τ being required in order to reach the minimum. Second, using binary pulsars with short orbital periods and small ratio of P_b/x one can make the minimum of $\sigma_v(\tau)$ as deep as possible in order to make the interval of stability of BPT scale as long as possible. We argue that determination of numerical value of minimum of $\sigma_v(\tau)$ from observations may give very likely more precise indicator of the upper limit on the energy density of stochastic gravitational radiation than measuring the slope of function $\sigma_z(\tau)$. The matter is that determination of minimum of $\sigma_v(\tau)$ is more statistically reliable than one of the slope of $\sigma_z(\tau)$ which depends to a large extent on uncertainty in calculation of errors of measurement of the curve on long time intervals (Kaspi *et al.* 1994).

9 CONCLUSIONS

We have done analytic calculations of time dependence of residuals and covariance matrix of fitting parameters of a binary pulsar under assumption that observations are evenly spaced. The results have been used to estimate the observed stability of intrinsic rotational and orbital frequencies of the pulsar. It was shown that the dynamic time scale BPT may be more stable under certain circumstances than the kinematic PT scale. It was clarified that stability of BPT is restricted (like that of PT) by the presence of red noise even when the orbital motion of the pulsar can be considered as pure deterministic. Nevertheless, this restriction on stability of BPT is important only if observations are run on time intervals being significantly longer than characteristic time of instability of PT. This remarkable property of BPT leads us to hope that its usage may help in future to undertake deeper analysis of laws of gravitational physics. There is no doubt that long-term stability of BPT time scale makes it as a useful practical tool for fundamental metrology of time.

Investigation of problem of construction and maintanace of BPT scale for studing various problems of modern astronomy

relates tightly to the problem of identification of nature and spectral characteristic of timing noise presenting in TOA. There is a hope that publication of papers of Deshpande *et al.* (1996) and D'Alessandro *et al.* (1997) devoted to detailed discussion of timing noise in the case of a sample of 18 single pulsars will attract a new interest to this problem. In particular, a reader of these papers will find a handful cases where the slopes are possibly more consistent with the spectral indices $n = 3, 5$ etc. than say $n = 4, 6$. Of course, the uncertainty in some of these cases is large but still there is an indicative support for the flicker noise cases.

Theoretical development of ideas relating to the study of stability of BPT time scale can be continued in several directions. Probably, one should consider binary pulsars with elliptical orbits having maximal ratio P_b/x which can improve quality of the time scale and, consequently, testing General Relativity in the strong field regime. In this problem it will be necessary to take into account possible variations of orbital elements of the binary caused by different, not well predicted factors, like flybys of stars or gravitational waves with frequncies being close to the orbital one (Chicone *et al.* 1998). It is not so difficult to show that for the BPT it is preferable to take binary pulsars with small enough ratio m_p/m_c (m_p and m_c are masses of pulsar and its companion respectively), long semi-major axis of the pulsar's orbit a_p, short orbital period P_b, and $\sin i \simeq 1$, that is large $x = a_p \sin i/c$. It would be desirable and worthwhile to create a network of reference binary pulsars and start their continuous timing observations in leading radio astronomical observatories of the world in order to make the concept of BPT time scale practically available as soon as possible.

ACKNOWLEDGEMENTS

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